Course in Linear Algebra with Some Exercises

- ✓ Course Outline.
- ✓ Illustrative Exercises.
- ✓ Solved Exams.

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Preface

This handout was intended for students enrolled in the first year of science and technology in the LMD system, especially students in preparatory classes in science and technology for the second semester of the academic year. The content of this handout corresponds to the official program for the subject Algebra II taught in first year. The manuscript contains five chapters :
- Vector spaces.
- Linear applications.
- Matrix, determinants and associated matrices.
- Solving systems of linear equations.
- Matrix reductions.

At the end of each chapter, there is a set of exercises left for the reader to try to solve and test the extent of his understanding and assimilation of the concepts presented. In the last, we add a chapter and we proposed in it a set of exams and interrogations for students in recent years, with suggested solutions for them in order for the student to practice the typical method for answering exams.

Chapitre 1

Vector spaces

Throughout this course, we consider that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ a commutative field. After having "done geometry" in the plane ($\mathbb{R}^2$) or in space ($\mathbb{R}^3$) for centuries, mathematicians had the idea of trying to identify those properties of these spaces which in fact made it possible to "doing geometry". These basic properties are reduced to this : we can add two vectors and we can multiply a vector by an element of $\mathbb{K}$ "a scalar", these two operations obeying a few simple rules of compatibility.

1.1 Definitions

A vector space on $\mathbb{K}$ (we also say $\mathbb{K}$-vector space) is a non-empty set $\mathbb{E}$ provided with two composition laws :

– Law of Internal Composition, noted $+$:

$$\forall\,(u,v) \in \mathbb{E}\times\mathbb{E} : u+v \in \mathbb{E}.$$

– Law of External Composition on $\mathbb{K}$, denoted $\times$ or $\cdot$:

$$\forall\,(\alpha,u) \in \mathbb{K}\times\mathbb{E} : \alpha\times u \in \mathbb{E},$$

Such that,

1. $(\mathbb{E},+)$ is a commutative group, i.e.
   – $\forall u,v \in \mathbb{E} : u+v = v+u$,
   – $\forall u,v,w \in \mathbb{E} : (u+v)+w = u+(v+w)$,
   – The neutral element for $+$ in $\mathbb{E}$ is $0_{\mathbb{E}}$ the zero vector, $u+0_{\mathbb{E}} = u$,
   – $\forall u \in \mathbb{E}$ vecteur, $-u$ its inverse , $u+(-u) = 0_{\mathbb{E}}$.
2. $\forall\,(u,v) \in \mathbb{E}\times\mathbb{E}, \forall\alpha \in \mathbb{K} : \alpha\times(u+v) = \alpha\times u+\alpha\times v$,
3. $\forall u \in \mathbb{E}, \forall\,(\alpha,\beta) \in \mathbb{K}\times\mathbb{K} : (\alpha+\beta)\,u = \alpha\times u+\beta\times u$,
4. $\forall u \in \mathbb{E}, \forall\,(\alpha,\beta) \in \mathbb{K}\times\mathbb{K} : \alpha\times(\beta\times u) = (\alpha\times\beta)\times u$,
5. $\forall u \in \mathbb{E}, 1\times u = u$

- The elements of $\mathbb{E}$ are called vectors.
- The elements of $\mathbb{K}$ are called scalars.
- The triple $(\mathbb{E}, +, \times)$ is a vector spaces on $\mathbb{K}$.

Example 1.1.1. *:*

1. $\mathbb{R}^2$, $\mathbb{R}^3$ *are vector spaces on* $\mathbb{K}$.
2. *More generally* $(\mathbb{R}^n, +, \times)$ *is a vector space on* $\mathbb{R}$ *and* $\mathbb{C}$.
3. *The set* $(\mathbb{R}_n[X], +, \times)$ *is a* $\mathbb{R}$*-vector space.*
4. *Let* $\mathbb{E}$ *and* $\mathbb{F}$ *be two non-empty sets. The set of maps defined from* $\mathbb{E}$ *to* $\mathbb{E}$ *is a vector space on* $\mathbb{K}$.

- *Let* f *and* g *be two applications,* $\forall x \in \mathbb{E} : (f+g)(x) = f(x) + g(x)$,
- *Let* f *be an application,* $\forall x \in \mathbb{E}, \forall \alpha \in \mathbb{K} : (\alpha f)(x) = \alpha f(x)$.

1.2 Vector Subspaces :

Definition 1.2.1. *Let* $E\mathbb{E}$ *be a* $\mathbb{K}$*-vector space and* $\mathbb{H}$ *a non-empty subset of* $E\mathbb{E}$*. We say that* $\mathbb{H}$ *is a vector subspace of* $E\mathbb{E}$ *if and only if :*

1. $0_{\mathbb{E}} \in \mathbb{H}$,
2. $\forall\, (u, v) \in \mathbb{H} \times \mathbb{H} : u + v \in \mathbb{H}$,
3. $\forall\, (\lambda, u) \in \mathbb{K} \times \mathbb{H} : \lambda \times u \in \mathbb{H}$.

Example 1.2.1.

$\mathbb{H}_1 = \left\{(x, y) \in \mathbb{R}^2 : y = ax, a \in \mathbb{R}\right\}$ *is a real line of* $\mathbb{R}^2$*, it is a vector subspace.*
$\mathbb{H}_2 = \left\{(x, y) \in \mathbb{R}^2 : y = x - 1\right\}$ *is not subspace of* $\mathbb{R}^2$.
$\mathbb{H}_3 = \left\{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\right\}$ *is not subspace of* $\mathbb{R}^2$.
$\mathbb{H}_4 = \left\{(x, y) \in \mathbb{R}^2 : x = 0 \vee y = 0\right\}$ *is not subspace of* $\mathbb{R}^2$.
$\mathbb{H}_5 = \left\{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0; a, b, c \in \mathbb{R}\right\}$ *is plane of* $\mathbb{R}^3$*, it is a* $\mathbb{R}-$*vector subspace.*

1.3 Sum of vector subspaces

Definition 1.3.1. *Let* $\mathbb{F}$ *and* $\mathbb{G}$ *be two vector subspaces of a vector space* $\mathbb{E}$*, then we call the sum of* $\mathbb{F}$ *and* $\mathbb{G}$ *the subset* $\mathbb{F} + \mathbb{G}$ *of* $\mathbb{E}$ *defined by*

$$\mathbb{F} + \mathbb{G} = \{X + Y : X \in \mathbb{F}, Y \in \mathbb{G}\}.$$

Definition 1.3.2. *Let* $\mathbb{F}$ *and* $\mathbb{G}$ *two vector subspaces of* $\mathbb{E}$ *on* $\mathbb{K}$*. We say that* $\mathbb{F}$ *and* $\mathbb{G}$ *are in direct sum or that the sum of* $\mathbb{F}$ *and* $\mathbb{G}$ *is direct, denoted by* $\mathbb{F} \oplus \mathbb{G}$*, if and anly if*

$$\mathbb{F} \cap \mathbb{G} = \{0_{\mathbb{E}}\}.$$

Example 1.3.1. $\mathbb{D}_1$ *and* $\mathbb{D}_2$ *are two vector subspaces of* $\mathbb{R}^2$ *be defined by*

$$\begin{aligned}\mathbb{D}_1 &= \left\{(x,y)\in\mathbb{R}^2 : y=0\right\},\\ \mathbb{D}_2 &= \left\{(x,y)\in\mathbb{R}^2 : x=0\right\}.\end{aligned}$$

Then

$$\mathbb{D}_1\cap\mathbb{D}_2=\left\{(x,y)\in\mathbb{R}^2 : x=0 \text{ and } y=0\right\}=\{(0,0)\}=\{0_{\mathbb{R}^2}\}.$$

So

$$\mathbb{D}_1\oplus\mathbb{D}_2=\mathbb{R}^2.$$

Definition 1.3.3. *Let* $\mathbb{F}$ *and* $\mathbb{G}$ *two vector subspaces of* $\mathbb{E}$ *(*$\mathbb{E}$ *an* $\mathbb{R}-$*vector subspace). We say that* $\mathbb{F}$ *is the* ***supplement*** *of* $\mathbb{G}$*, if and anly if*

- $\mathbb{E}=\mathbb{F}+\mathbb{G}$,
- $\mathbb{F}\cap\mathbb{G}=\{0_{\mathbb{E}}\}$, *and denoted by*

$$\mathbb{E}=\mathbb{F}\oplus\mathbb{G}.$$

Proposition 1.3.1. *Let* $\mathbb{F}$ *and* $\mathbb{G}$ *be two vector subspaces of a vector space* $\mathbb{E}$*, show that*

1. $\mathbb{F}\cap\mathbb{G}$ *is a vector subspace.*
2. $\mathbb{F}+\mathbb{G}$ *is a vector subspace.*
3. *If* $\mathbb{F}\oplus\mathbb{G}=\mathbb{E}$*, then*

$$\forall w\in\mathbb{E},\exists! u\in\mathbb{F},\exists! v\in\mathbb{G}:\ w=u+v.$$

Proof. 1. $\mathbb{F}\cap\mathbb{G}$ is a vector subspace.

(a) $0\in\mathbb{F}\cap\mathbb{G}$?

As $\mathbb{F}$ and $\mathbb{G}$ are vector subspaces, then

$$\left\{\begin{array}{l}0\in\mathbb{F}\\ 0\in\mathbb{G}\end{array}\right.\Longrightarrow 0\in\mathbb{F}\cap\mathbb{G}.$$

(b) Let $u,v\in\mathbb{F}\cap\mathbb{G}$, then is it that $u+v\in\mathbb{F}\cap\mathbb{G}$?

$$\left\{\begin{array}{l}u\in\mathbb{F}\cap\mathbb{G}\\ v\in\mathbb{F}\cap\mathbb{G}\end{array}\right.\Longrightarrow\left\{\begin{array}{l}u\in\mathbb{F}\wedge u\in\mathbb{G}\\ v\in\mathbb{F}\wedge v\in\mathbb{G}\end{array}\right..$$

Since $\mathbb{F}$ and $\mathbb{G}$ are subspaces, then

$$\left\{\begin{array}{l}u+v\in\mathbb{F}\\ u+v\in\mathbb{G}\end{array}\right.\Longrightarrow u+v\in\mathbb{F}\cap\mathbb{G}.$$

(c) Let $u\in\mathbb{F}\cap\mathbb{G}$ and $\lambda\in\mathbb{R}$, then is it that $\lambda u\in\mathbb{F}\cap\mathbb{G}$?

$$\left\{\begin{array}{l}\lambda u\in\mathbb{F}\\ \lambda u\in\mathbb{G}\end{array}\right.\Longrightarrow\lambda u\in\mathbb{F}\cap\mathbb{G}.$$

2. $\mathbb{F}+\mathbb{G}$ is a vector subspace.

   (a) $0 \in \mathbb{F} \cap \mathbb{G}$?

   As $\mathbb{F}$ and $\mathbb{G}$ are vector subspaces, then

$$\left\{ \begin{array}{l} 0 \in \mathbb{F} \\ 0 \in \mathbb{G} \end{array} \right. \Longrightarrow 0 = \underbrace{0}_{\in \mathbb{F}} + \underbrace{0}_{\in \mathbb{G}} \in \mathbb{F}+\mathbb{G}.$$

   (b) Let $w, z \in \mathbb{F}+\mathbb{G}$, then is it that $w+z \in \mathbb{F}+\mathbb{G}$?

$$\left\{ \begin{array}{l} w \in \mathbb{F}+\mathbb{G} \\ z \in \mathbb{F}+\mathbb{G} \end{array} \right. \Longrightarrow \left\{ \begin{array}{c} \exists u \in \mathbb{F}, \exists v \in \mathbb{G} : w = u+v \\ \exists u' \in \mathbb{F}, \exists v' \in \mathbb{G} : z = u'+v' \end{array} \right. .$$

   Since $\mathbb{F}$ and $\mathbb{G}$ are subspaces, then

$$\left\{ \begin{array}{l} u+u' \in \mathbb{F} \\ v+v' \in \mathbb{G} \end{array} \right. \Longrightarrow w+z = u+u'+v+v' \in \mathbb{F}+\mathbb{G}.$$

   (c) Let $w \in \mathbb{F}+\mathbb{G}$ and $\lambda \in \mathbb{R}$, then is it that $\lambda w \in \mathbb{F}+\mathbb{G}$?

$$w \in \mathbb{F}+\mathbb{G} \Rightarrow w = u+v, u \in \mathbb{F}, v \in \mathbb{G},$$

   then

$$\left\{ \begin{array}{l} \lambda u \in \mathbb{F} \\ \lambda v \in \mathbb{G} \end{array} \right. \Longrightarrow \lambda w = \lambda u + \lambda v \in \mathbb{F}+\mathbb{G}.$$

3. $\mathbb{F} \oplus \mathbb{G} = \mathbb{E}$, then

$$\mathbb{E} = \mathbb{F}+\mathbb{G}, \text{ and } \mathbb{F} \cap \mathbb{G} = \{0_{\mathbb{E}}\}.$$

   That's to say,

$$w \in \mathbb{E} \Rightarrow \exists u \in \mathbb{F}, \exists v \in \mathbb{G} : w = u+v.$$

   It reste to demonstrate uniqueness. Suppose absurdly that there exist $u \in \mathbb{F}$ and $v \in \mathbb{G}$. So

$$\begin{array}{rcl} (u+v = u'+v') & \Rightarrow & u-u' = v'-v \\ & \Rightarrow & \underbrace{u-u'}_{\in \mathbb{F}}, \underbrace{v'-v}_{\in \mathbb{G}} \in \mathbb{F} \cap \mathbb{G} = \{0_{\mathbb{E}}\} \\ & \Rightarrow & u-u' = v'-v = 0_{\mathbb{E}} \end{array}$$

□

1.4 Subspace generated

Definition 1.4.1. *Let $\mathbb{F}$ a vector subspace of a vector space $\mathbb{E}$ an $\mathbb{R}-$and $v_1, v_2, ..., v_p \in \mathbb{E}$. We call the space generated by $v_1, v_2, ..., v_p$, the vector subspace of $\mathbb{E}$ denoted $\mathcal{L}\{v_1, v_2, ..., v_p\}$ or $\langle v_1, v_2, ..., v_p \rangle$ defined by*

$$\langle v_1, v_2, ..., v_p \rangle = \left\{ \sum_{i=1}^{p} \lambda_i v_i : \lambda_1, \lambda_2, ..., \lambda_p \in \mathbb{R} \right\}.$$

Example 1.4.1. *Let the subset $\mathbb{F}$ of $\mathbb{R}^3$ be defined by*

$$\begin{aligned}
\mathbb{F} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3; x + y + z = 0 \right\} \\
&= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3; x = -y - z \right\} \\
&= \left\{ \begin{pmatrix} -y - z \\ y \\ z \end{pmatrix}; y, z \in \mathbb{R} \right\} \\
&= \left\{ y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}; y, z \in \mathbb{R} \right\} \\
&= \left\langle \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle.
\end{aligned}$$

1.5 Families of vector spaces

1.5.1 Linearly Independent (L.I) and Linearly Dependent (L.D)

Definition 1.5.1. *A family $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ of a vector space $\mathbb{E}$ is said to be **Linearly independent (Free)** if and only if*

$$\forall (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i v_i = 0_{\mathbb{E}} \Rightarrow \lambda_1 = \lambda_2 = ... = \lambda_n = 0.$$

*A family $\mathcal{B}$ is said to be **Linearly dependent (Linked)**, if and only if $\exists (\lambda_1, \lambda_2, ..., \lambda_n) \neq (0, 0, ..., 0)$ such that*

$$\sum_{i=1}^{n} \lambda_i v_i = 0_{\mathbb{E}}.$$

Example 1.5.1. *:*

1- Let the family $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$,

Let us show that $\mathcal{B} = \{v_1, v_2\}$ *form a Linearly independent family the vector subspace of the previous example* $\mathbb{F}$*.*
Let $\alpha_1, \alpha_2 \in \mathbb{R}$*, then*

$$\alpha_1 v_1 + \alpha_2 v_2 = O_{\mathbb{R}^3} \Leftrightarrow \begin{cases} -\alpha_1 - \alpha_2 = 0 \\ \alpha_1 = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0.$$

2- The family $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}, v_2 = \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ *form a linearly independent family of* $\mathbb{R}^3$*.*
Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$*, then*

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = O_{\mathbb{R}^3} \Leftrightarrow \begin{cases} \alpha_1 - 5\alpha_2 = 0 \\ -\alpha_1 + 2\alpha_2 = 0 \\ 7\alpha_1 + 3\alpha_2 + \alpha_3 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_1 = \alpha_3 = 0.$$

3- The family $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$ *is not linearly independent (linearly dependent) family of* $\mathbb{R}^3$ *because* $v_3 = v_1 + v_2$*.*

1.5.2 Basis of a vector space

Definition 1.5.2. *A family* $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ *of vectors of a vector space* $\mathbb{E}$ *is said to be a* ***basis*** *if and only if the following two conditions are verified*

1. $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ *is a linearly independent family*
2. *The family* $\mathcal{B}$ *is called generator of* $\mathbb{E}$ *i.e*
   $\forall u \in \mathbb{E}, \exists\, (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}^n$*, such that* $u = \sum_{i=1}^{n} \alpha_i v_i$.

Example 1.5.2. *:*

1- The family $\mathcal{B} = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ *is a canonical basis of* $\mathbb{R}^3$.

2- Let the family $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$*, form a basis of the vector subspace of the previous example* $\mathbb{F}$*.*

3- The family $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix}, v_2 = \begin{pmatrix} -5 \\ 2 \\ 3 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ *form a basis of* $\mathbb{R}^3$.

4- The family $\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$ *is not basis of* $\mathbb{R}^3$ *(linearly dependent).*

Remark 1.5.1. *1. The canonical basis of* $\mathbb{R}^n$ *is the family*

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}.$$

2. The canonical basis of $\mathbb{R}_n[X]$ *is the family* $\{1, X, X^2, \cdots, X^n\}$.

1.5.3 Dimension of a vector space

Definition 1.5.3. *We say that a vector space* $\mathbb{E}$ *is of dimension* $n \in \mathbb{N}$ *and we write* $\dim \mathbb{E} = n$ *if there exists in* $\mathbb{E}$ *a basis containing* n *vectors.*

Example 1.5.3. $\dim \mathbb{R}_n[X] = n+1, \dim \mathbb{R}^n = n, \dim \{0\} = 0,$ *by convention.*

Remark 1.5.2. $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ *is a basis of* $\mathbb{E}$ *iff* $\mathcal{B}$ *is a linearly independent family and* $card\mathcal{B} = \dim \mathbb{E}$.

Definition 1.5.4. *A family* $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ *of vectors of a vector space* $\mathbb{E}$ *is said to be a* ***basis*** *if and only if the following two conditions are verified*

1. $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ *is a linearly independent family*
2. $\dim \mathbb{E} = card\mathcal{B}$.

Definition 1.5.5. *Let* $\mathbb{F}$ *and* $\mathbb{G}$ *two vector subspaces of* $\mathbb{E}$ *(*$\mathbb{E}$ *an* $\mathbb{R}-$*vector subspace), with* $\mathbb{F}$ *is the supplement of* $\mathbb{G}$*, then we have*

$$\mathbb{E} = \mathbb{F} \oplus \mathbb{G} \Leftrightarrow \mathbb{F} \cap \mathbb{G} = \{0_{\mathbb{E}}\}, and \dim \mathbb{F} + \dim \mathbb{G} = \dim \mathbb{E}.$$

1.6 Exercises set

Exercise 1.6.1. *:*
Let $\mathbb{E} = \mathbb{R}_+^*$, $\oplus$ *and* $\otimes$ *be two laws of internal and external composition respectively such as :*

$$\forall a, b \in \mathbb{E} \quad and \quad \forall \lambda \in \mathbb{R}; \quad a \oplus b = a \cdot b \quad et \quad \lambda \otimes a = a^{\lambda}.$$

Show that $(\mathbb{E}, \oplus, \otimes)$ *is a vector space on* $\mathbb{R}$.

Exercise 1.6.2. *:*
Let $\mathbb{E} = \mathbb{R}^2$. *We define the following two laws on* $\mathbb{E}$:

$$\forall (x, y), (x', y') \in \mathbb{E} \quad et \quad \forall \lambda \in \mathbb{R};$$
$$(x, y) \circledast (x', y') = (x + x', y + y') \quad and \quad \lambda \odot (x, y) = (\lambda x, 0).$$

Is $(\mathbb{E}, \circledast, \odot)$ *a vector space on* $\mathbb{R}$ *?*

Exercise 1.6.3. *:*
From the the following sets $\mathbb{E}_1$, $\mathbb{E}_2$, $\mathbb{E}_3$, $\mathbb{E}_4$ *and* $\mathbb{E}_5$, *determine which are vector subspaces of* $\mathbb{E}$ *:*

$$\begin{aligned}
\mathbb{E}_1 &= \left\{(x, y, z) \in \mathbb{R}^3; \quad x = 2y \quad and \quad z = 0\right\}, \quad \mathbb{E} = \mathbb{R}^3, \\
\mathbb{E}_2 &= \left\{(x, y, z) \in \mathbb{R}^3 : -x + 3y - 7z = 0\right\}, \mathbb{E} = \mathbb{R}^3, \\
\mathbb{E}_3 &= \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\right\}, \quad \mathbb{E} = \mathbb{R}^2, \\
\mathbb{E}_4 &= \left\{(x, y, z, t) \in \mathbb{R}^4 : -x + z + t = 0 \ and \ x - y + z = 0\right\}, \quad \mathbb{E} = \mathbb{R}^4, \\
\mathbb{E}_5 &= \left\{P \in \mathbb{R}_2[X] : 2P'' - XP = 0\right\}, \quad \mathbb{E} = \mathbb{R}_2[X].
\end{aligned}$$

Exercise 1.6.4. *:*
From the following families, determine which are linearly independent then which are basis of $\mathbb{E}$ *:*

$$\begin{aligned}
\mathcal{B}_1 &= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}, \mathbb{E} = \mathbb{R}^3. \\
\mathcal{B}_2 &= \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \right\}, \mathbb{E} = \mathbb{R}^3. \\
\mathcal{B}_3 &= \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \mathbb{E} = \mathbb{R}^4. \\
\mathcal{B}_4 &= \left\{X^2 + X - 1, 1 - X^2, X + 1^2\right\}, \mathbb{E} = \mathbb{R}_2[X]. \\
\mathcal{B}_5 &= \left\{X^3 - 2X^2 + 1, 1 + X^3, 1 - X + 3X^3, X + X^2\right\}, \mathbb{E} = \mathbb{R}_3[X].
\end{aligned}$$

Exercise 1.6.5. *:*

Let $\mathbb{E}$ and $\mathbb{F}$ be two vector spaces on the same field.
The Cartesian product $\mathbb{E} \times \mathbb{F}$ is also a vector space over $\mathbb{K}$ for the both laws :
$\forall (x,y),(x',y') \in \mathbb{E} \times \mathbb{F}$ *and* $\forall \lambda \in \mathbb{K}$;

$$(x,y)+(x',y') = (x+x', y+y') \quad \text{and} \quad \lambda \cdot (x,y) = (\lambda x, \lambda y).$$

Let $\{u_1, u_2, ..., u_n\}$ be a basis of $\mathbb{E}$ and let $\{v_1, v_2, ..., v_m\}$ be a basis of $\mathbb{F}$.

1. *Show that the family $\{(u_1,0),(u_2,0),...,(u_n,0),(0,v_1),(0,v_2),...,(0,v_m)\}$ is a basis of $\mathbb{E} \times \mathbb{F}$.*
2. *Deduce that :* $\dim(\mathbb{E} \times \mathbb{F}) = \dim \mathbb{E} + \dim \mathbb{F}$.

Exercise 1.6.6. *We consider in $\mathbb{R}^4$ the vectors :*

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, v_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Let $\mathbb{E}$ be the vector subspace generated by v_1, v_2, v_3 and $\mathbb{F}$ that generated by v_4, v_5.

1. *Calculate the dimensions of $\mathbb{E}$,$\mathbb{F}$,$\mathbb{E} \cap \mathbb{F}$ and $\mathbb{E} + \mathbb{F}$.*
2. *Do we have $\mathbb{E} \oplus \mathbb{F} = \mathbb{R}^4$?.*

Exercise 1.6.7. *In $\mathbb{R}^3$, we consider the two vector subspaces.*

$$\begin{aligned} \mathbb{E}_1 &= \left\{(x,y,z) \in \mathbb{R}^3; x + ay + z = 0,\ a \in \mathbb{R}\ \right\}, \\ &\quad \text{and} \\ \mathbb{E}_2 &= \left\{(x,y,z) \in \mathbb{R}^3; x = -y = z\right\}. \end{aligned}$$

For what values of a we have : $\mathbb{R}^3 = \mathbb{E}_1 \oplus \mathbb{E}_2$.

Chapitre 2

Linear applications

In mathematics, a linear application is an application between two vector spaces that respects the addition of vectors and scalar multiplication, and thus preserves linear combinations more generally. The expression can also be used for a morphism between two modules on a ring, with a similar presentation apart from the notions of base and dimension.

2.1 Definitions

Definition 2.1.1. *(Linear application). Let $\mathbb{E}$ and $\mathbb{F}$ be two vectors spaces. We say that an application f*

$$f : \mathbb{E} \to \mathbb{F}$$

is linear if and only if

1. $\forall u, v \in \mathbb{E} : f(u+v) = f(u) + f(v)$,
2. $\forall u \in \mathbb{E}, \forall \alpha \in \mathbb{R} : f(\alpha u) = \alpha f(u)$.

Remark 2.1.1.
- *The space of all linear application from $\mathbb{E}$ to $\mathbb{F}$ is a vectors space denoted by $\mathcal{L}(\mathbb{E}, \mathbb{F})$.*
- *If $f(0_{\mathbb{E}}) \neq 0_{\mathbb{F}}$, then f is not linear application.*
- *Conditions 1 and 2 can be combined as follows :*

$$\forall u, v \in \mathbb{E}, \forall \alpha \in \mathbb{R} : f(u + \alpha v) = f(u) + \alpha f(v).$$

Example 2.1.1. *Let the application*

$$f : \mathbb{R}^3 \to \mathbb{R}^2, f(x, y, z) = (-2x, y + 3z),$$

we show that f is linear. Let $u=(x,y,z), v=(x',y',z') \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} f(u+v) &= f\left(x+x', y+y'^{,z+z'}\right) \\ &= \left(-2\left(x+x'\right), y+y'+3\left(z+z'\right)\right) \\ &= (-2x, y+3z)+(-2x', y'+3z') \\ &= f(u)+f(v), \end{aligned}$$

and

$$\begin{aligned} f(\alpha u) &= f\left(\alpha x, \alpha y\right) \\ &= \left(-\left(\alpha x\right), \alpha y+3\left(\alpha z\right)\right) \\ &= \alpha f(u). \end{aligned}$$

Proposition 2.1.1. *Let $\mathbb{E}$ and $\mathbb{F}$ be two vectors spaces, $f:\mathbb{E}\to\mathbb{F}$ is a linear application. Then*

1. $f(0_{\mathbb{E}})=0_{\mathbb{F}}$.
2. $f(-u)=-f(u)$.

2.2 Kernel and Image of the linear application.

Definition 2.2.1. *(Kernel of a linear application.) Let $\mathbb{E}$ and $\mathbb{F}$ be two vectors spaces and $f:\mathbb{E}\to\mathbb{F}$ is a linear application.*
We call the kernel of the linear application f the subset denoted $\ker f$ *defined as follows :*

$$\ker f = \{u\in\mathbb{E} : f(u)=0_{\mathbb{F}}\}\subset\mathbb{E}.$$

Definition 2.2.2. *(Image of a linear application.) Let $\mathbb{E}$ and $\mathbb{F}$ be two vectors spaces and $f:\mathbb{E}\to\mathbb{F}$ is a linear application.*
We call the image of the linear application f the subset denoted Imf defined as follows :

$$Imf = \{Y\in\mathbb{F}, \exists X\in\mathbb{E} : f(X)=Y\} = \{f(X) : X\in\mathbb{E}\}\subset\mathbb{F}.$$

Example 2.2.1. *Let $f:\mathbb{R}^3\to\mathbb{R}^3$ be a linear application defined by*

$$f(x,y,z)=\left(x-y, y-z, x+2z\right).$$

$$\begin{aligned} \ker f &= \left\{u\in\mathbb{R}^3 : f(u)=0_{\mathbb{R}^3}\right\} \\ &= \left\{(x,y,z)\in\mathbb{R}^3 : (x-y, y-z, x+2z)=(0,0,0)\right\}. \end{aligned}$$

Then

$$(x-y, y-z, x+2z) = (0,0,0) \Leftrightarrow \begin{cases} x-y=0 \\ y-z=0 \\ x+2z=0 \end{cases} \Rightarrow (x,y,z) = (0,0,0).$$

So

$$\ker f = \{0_{\mathbb{R}^3}\}.$$

$$\begin{aligned} Imf &= \left\{f(X); X \in \mathbb{R}^3\right\} \\ &= \{(x-y, y-z, x+2z); x, y, z \in \mathbb{R}\} \\ &= \{x(1,0,1) + y(-1,1,0) + z(0,-1,2); x, y, z \in \mathbb{R}\} \\ &= \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\rangle \Rightarrow \dim Imf = 3 \\ &= \mathbb{R}^3. \end{aligned}$$

Proposition 2.2.1. *Let $\mathbb{E}$ and $\mathbb{F}$ be two vectors spaces, $f : \mathbb{E} \to \mathbb{F}$ and $g : \mathbb{F} \to \mathbb{G}$ are linear applications. Then*

1. *$\alpha f + \beta g$ and gof are linear applications, where $\alpha, \beta \in \mathbb{K}$.*
2. *$\ker f$ is a subspace of $\mathbb{E}$ and Imf is a subspace of $\mathbb{F}$.*

Proof. 1. Show that $\alpha f + \beta g, gof$ are linaer applications, where $\alpha, \beta \in \mathbb{K}$.
Let $u, v \in \mathbb{E}$,$\lambda \in \mathbb{K}$, then

–

$$\begin{aligned} (\alpha f + \beta g)(u + \lambda v) &= (\alpha f)(u + \lambda v) + (\beta g)(u + \lambda v) \\ &= \alpha(f(u + \lambda v)) + \beta(g(u + \lambda v)) \\ &= \alpha(f(u) + \lambda f(v)) + \beta(g(u) + \lambda g(v)) \\ &= \alpha f(u) + \beta g(u) + \lambda(\alpha f(v) + \beta g(v)) \\ &= (\alpha f + \beta g)(u) + \lambda(\alpha f + \beta g)(v). \end{aligned}$$

–

$$\begin{aligned} (gof)(u + \lambda v) &= g[f(u + \lambda v)] \\ &= g[f(u) + \lambda f(v)] \\ &= g[f(u)] + g[\lambda f(v)] \\ &= g[f(u)] + \lambda g[f(v)] \\ &= gof(u) + \lambda gof(v). \end{aligned}$$

2. Show that $\ker f$ is a subspace in $\mathbb{E}$. ($\ker f = \{u \in \mathbb{E} : f(u) = 0_{\mathbb{F}}\}$)

– $0_{\mathbb{E}} \in \ker f$, because f is linear application and $f(0_{\mathbb{E}}) = 0_{\mathbb{F}}$.
– Let $u, u' \in \ker f, \lambda \in \mathbb{K}$, then

$$\begin{aligned}[f(u) = 0_{\mathbb{F}} \text{ and } f(u') = 0_{\mathbb{F}}] &\Rightarrow [f(u) = 0_{\mathbb{F}} \text{ and } \lambda f(u') = \lambda 0_{\mathbb{F}} = 0_{\mathbb{F}}] \\ &\Rightarrow f(u) + \lambda f(u') = 0_{\mathbb{F}} + 0_{\mathbb{F}} = 0_{\mathbb{F}} \\ &\Rightarrow f(u) + \lambda f(u') = f(u) + f(\lambda u') = f(u + \lambda u') = 0_{\mathbb{F}} \\ &\Rightarrow u + \lambda u' \in \ker f.\end{aligned}$$

3. Show that Imf is a subspace in $\mathbb{F}$.
   – $0_{\mathbb{F}} \in Imf$, because f is linear application and $\exists 0_{\mathbb{E}} \in \mathbb{E}\ f(0_{\mathbb{E}}) = 0_{\mathbb{F}}$.
   – Let $v, v' \in Imf, \lambda \in \mathbb{K}$, then

$$v, v' \in Imf \Leftrightarrow \exists u, u' \in \mathbb{E} : v = f(u) \text{ and } v' = f(u'),$$

   so

$$v + \lambda v' = f(u) + \lambda f(u') = f(u + \lambda u').$$

   Then $\exists w = u + \lambda u' \in \mathbb{E}$ such that $v + \lambda v' = f(w) \Rightarrow v + \lambda v' \in Imf$.

□

2.2.1 Rank of a linear application.

Definition 2.2.3. *Let $\mathbb{E}$ and $\mathbb{F}$ be two vectors spaces and f is a linear application $f : \mathbb{E} \to \mathbb{F}$. We call the **rank** of the application f the dimension of its image and we write*

$$ran\,(f) = \dim Imf.$$

Theorem 2.2.2. *(Rank Theorem). Let $\mathbb{E}$ and $\mathbb{F}$ be two vectors spaces and f is a linear application $f : \mathbb{E} \to \mathbb{F}$. Then, we have*

$$\dim \mathbb{E} = \dim (\ker f) + ran(f).$$

Example 2.2.2. *Let the application*

$$f :\ \mathbb{R}^3 \to \mathbb{R}^2, f(x, y, z) = (-2x, y + 3z),$$

Determine the $ran(f)$:

$$\ker f \ = \ \left\{(x, y, z) \in \mathbb{R}^3 : (-2x, y + 3z) = (0, 0)\right\}.$$

Then

$$(-2x, y + 3z) = (0, 0) \Leftrightarrow \left\{ \begin{array}{c} -2x = 0 \\ y + 3z = 0 \end{array} \right. \Rightarrow x = 0 \wedge y = -3z.$$

So

$$\ker f = \left\langle \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} \right\rangle \Rightarrow \dim \ker f = 1.$$

According to the rank theorem,

$$\dim \mathbb{R}^3 = 1 + ran(f) \Rightarrow ran(f) = 2.$$

2.3 Injective, surjective linear applications

Proposition 2.3.1. *Let $\mathbb{E}$ and $\mathbb{F}$ be two vectors spaces and $f : \mathbb{E} \to \mathbb{F}$ is a linear application. Then*

1. *f is injective if and only if $\ker f = \{0_{\mathbb{E}}\}$.*
2. *T is surjective if and only if $ImT = \mathbb{F}$.*
3. *Suppose f is injective, then if $\{e_1, e_2, \cdots, e_n\}$ is a family linearly independent of $\mathbb{E}$, we will have $\{f(e_1), f(e_2), \cdots, f(e_n)\}$ is the family linearly independent of $\mathbb{F}$.*
4. *If f is bijective, then the reciprocal application f^{-1} is also linear.*

2.4 Exercises set

Exercise 2.4.1. *:*
Show that the following applications T are linear, then calculate $\ker T$ and ImT, deduce the injectivity, surjectivity of T and the rank of T.

1. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3; \quad T(x, y) = (2x + 3y, -x + 2y, 3x - 5y).$
2. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2; \quad T(x, y; z) = (2x + y + z, x + y + 2z).$
3. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3; \quad T(x, y) = (-2x + y - z, 3x + 3y - z, x - 2y + 2z).$
4. $T : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]; \quad T(P) = (1 - X)P' + P.$

Exercise 2.4.2. *:*
Let the application $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by :

$$f(x, y) = (x + y, x - y).$$

1. *Show that f is linear.*
2. *Determine $Ker(f)$ and give its dimension.*
3. *Give the basis and dimension of $Im(f)$.*
4. *f is it bijective ?*
5. *Determine $f \circ f$.*

Exercise 2.4.3. *:*
Let the linear application $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ *be defined for all* $X = (x, y, z, t)$ *by :*

$$f(x, y, z, t) = (x - y + z, 0, x + y - z + t, t).$$

Let $\mathbb{E} = \{(x, y, z, t) \in \mathbb{R}^4; \quad x + y - z + t = 0\}$.

1. *Give the basis and dimension of* $\ker(f)$.
2. *Give the basis and dimension of* $Im(f)$.
3. *Do we have* $\ker(f) \oplus Im(f) = \mathbb{R}^4$?
4. *Show that* $\mathbb{E}$ *is a vector subspace of* $\mathbb{R}^4$, *give a basis of* $\mathbb{E}$.
5. *Do we have* $\ker(f) \oplus \mathbb{E} = \mathbb{R}^4$?

Exercise 2.4.4. *:*
Let the application $T : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$ *be defined by :*

$$T(P) = P - P'.$$

1. *Show that* T *is linear.*
2. *Determine* $Ker(T)$ *and* $dimKer(T)$.
3. *Give the basis and dimension of* $Im(T)$.
4. T *is it injective ? sujective ? bijective ?*

Exercise 2.4.5. *:*
Let $\mathbb{E}$ *be a vector space of dimension* 3, $\{e_1, e_{2,3}\}$ *a basis of* $\mathbb{E}$ *and* t *a real parameter.*

1. *Demonstrate that the data of*

$$\begin{cases} g(e_1) = e_1 + e_{2,3} \\ g(e_2) = e_1 - e_2 \\ g(e_3) = e_1 + te_3 \end{cases}$$

defines a linear application g *from* $\mathbb{E}$ *to* $\mathbb{E}$.

2. *Write the transform of the vector* $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$.
3. *How to choose* t *so that* g *is injective ? surjective ?*

Chapitre 3

Matrix, determinants and associated matrices

Matrices are essential objects of linear algebra, which are useful in particular in solving systems of equations and for carrying out transformations on vectors. In addition, matrices have very powerful applications in many fields. In this chapter we will see a general notions about matrix calculation and the relationship between matrices and linear applications. We take note that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

3.1 Generalities

Definition 3.1.1. *A matrix over $\mathbb{K}$ is a **table (rectangular array)** of numbers of $\mathbb{K}$ defined by **rows (line)** and **columns**. In a matrix A of n rows and m columns defined as follows*

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nm} \end{pmatrix} = (a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq m}}.$$

The a_{ij} are the elements located on the i^{th} row and the j^{th} column.

Notation 3.1.1. *The set of matrices of n rows and m columns denoted by $\mathcal{M}_{n,m}(\mathbb{K})$. If $n = m$, it is denoted by $\mathcal{M}_n(\mathbb{K})$.*

Example 3.1.1. *1. Let a matrix $\mathcal{A}$*

$$\mathcal{A} = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 5 & -2 \end{pmatrix} = (a_{ij})_{\substack{1\leq i\leq 2 \\ 1\leq j\leq 3}}$$

2. The matrix $\mathcal{A}$ has two rows and three colums.

3. *The first row has elements* $2, -1$ *and* 0*, the third column has elements* 0 *and* -2*.*

4. *The number* 3 *is located in the row two and column one.*

3.1.1 Useful notations

3.1.1.1 Square matrix

– A **square matrix** whose number of its rows is equal to the number of its columns $(n = m)$.

Example 3.1.2.

$$A = \begin{pmatrix} 5 & -11 \\ 10 & 20 \end{pmatrix}, B = \begin{pmatrix} 0 & 2 & -5 \\ 1 & 0 & 7 \\ 1 & 4 & -9 \end{pmatrix}$$

3.1.1.2 Main diagonal

– The **main diagonal** of a matrix $\mathcal{A}$ are the elements a_{ij} where $i = j$ (i.e a_{11}, a_{22}, $a_{33}, \cdots$).

Example 3.1.3.

$$\mathcal{A} = \begin{pmatrix} 1 & 4 & 8 \\ 11 & -1 & 1 \\ 3 & 7 & 15 \end{pmatrix},$$

the elements of main diagonal of a matrix $\mathcal{A}$ *are* $\{1, -1, 15\}$

3.1.1.3 Diagonal matrix

– A **diagonal matrix** is a square matrix and the non-main diagonal elements are zero, i.e.

$$a_{ij} = 0, \text{for all } i \neq j.$$

Example 3.1.4.

$$A = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix}, B = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$$

3.1.1.4 Identity matrix

– **Identity matrix** is a diagonal matrix and all its elements of diagonal equal to one, i.e. $a_{ii} = 1$ and, $a_{ij} = 0$, for all $i \neq j$. It is denoted by I_n, such that

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Example 3.1.5.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.1.1.5 Zero matrix (Null matrix)

– **Zero matrix (Null matrix)** is a matrix with all of its elements are zero.

Example 3.1.6.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3.1.1.6 Triangular matrix

3.1.1.6.1 Upper triangular matrix

– **Upper triangular matrix** is when all elements **below** the main daigonal are zero

$$a_{ij} = 0, \text{for all } i > j.$$

Example 3.1.7.

$$A = \begin{pmatrix} -4 & 0 & -1 \\ 0 & 10 & 4 \\ 0 & 0 & 5 \end{pmatrix}, B = \begin{pmatrix} -1 & 6 \\ 0 & 2 \end{pmatrix}$$

3.1.1.6.2 Lower triangular matrix

– **Lower triangular matrix** is when all elements **above** the main daigonal are zero

$$a_{ij} = 0, \text{for all } i < j.$$

Example 3.1.8.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 9 & 14 & 0 \\ -8 & 6 & 0 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ 4 & -9 \end{pmatrix}$$

3.1.1.7 Dimensions of matrix

– The **dimensions** of a matrix of n rows and p columns are (n, p) and we denote $n \times p$.

3.2 Matrix operations

3.2.1 Matrix additon

Definition 3.2.1. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices in $\mathcal{M}_{n,m}(\mathbb{K})$: The sum of A and B, denoted $A + B$, is*

$$A + B = (a_{ij} + b_{ij}) = \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1p} + b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & \cdots & a_{np} + b_{nm} \end{pmatrix}.$$

3.2.2 Scalar multiplication

Definition 3.2.2. *Let $A = (a_{ij})$ be a matrix in $\mathcal{M}_{n,m}(\mathbb{K})$ and let α be a scalar : The scalar multiplication of α and A, denoted αA, is*

$$\alpha A = (\alpha a_{ij}) = \begin{pmatrix} \alpha a_{11} & \cdots & \alpha a_{1m} \\ \vdots & \ddots & \vdots \\ \alpha a_{n1} & \cdots & \alpha a_{nm} \end{pmatrix}.$$

Proposition 3.2.1. *The space $(\mathcal{M}_{n,p}(\mathbb{K}), +, \times)$ is a **vector space** on $\mathbb{K}$).*

Example 3.2.1. *let*

$$A = \begin{pmatrix} 4 & -6 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} -5 & 3 \\ -1 & 8 \end{pmatrix}, C = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 3 & 4 \end{pmatrix},$$

1. $2A + B = \begin{pmatrix} 3 & -9 \\ 1 & 8 \end{pmatrix}$.
2. $A + C, B + C$ *are not defined.*
3. $B - A = \begin{pmatrix} -9 & 12 \\ -2 & 8 \end{pmatrix}$

3.2.3 Matrix multiplication

Definition 3.2.3. *Let $A = (a_{ij}) \in \mathcal{M}_{n,p}(\mathbb{K})$ and $B = (b_{ij}) \in \mathcal{M}_{p,m}(\mathbb{K})$ be two matrices. The matrix product of A and B, denoted $A.B$, or simply AB, is the matrix $\mathcal{C} = (c_{ij}) \in \mathcal{M}_{n,m}(\mathbb{K})$ where c_{ij} is obtained by the product of the i^{th} row of A and the j^{th} column of B.*

$$\mathcal{C} = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{a}_{i1} & \cdots & \mathbf{a}_{ij} & \cdots & \mathbf{a}_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{np} \end{pmatrix}}_{n \times p} \times \underbrace{\begin{pmatrix} b_{11} & \cdots & \mathbf{b}_{1j} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ b_{i1} & \cdots & \mathbf{b}_{ij} & \cdots & b_{im} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{p1} & \cdots & \mathbf{b}_{pj} & \cdots & b_{pm} \end{pmatrix}}_{}.$$

Definition 3.2.4. *Then the element (c_{ij}) is the element defined as follows*

$$c_{ij} = \begin{pmatrix} \mathbf{a}_{i1} & \cdots & \mathbf{a}_{ij} & \cdots & \mathbf{a}_{ip} \end{pmatrix} \times \begin{pmatrix} \mathbf{b}_{1j} \\ \vdots \\ \mathbf{b}_{ij} \\ \vdots \\ \mathbf{b}_{pj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$$

Remark 3.2.1. *$A \in \mathcal{M}_{n,p}(\mathbb{K})$ and $B \in \mathcal{M}_{p,m}(\mathbb{K}) \Rightarrow AB \in \mathcal{M}_{n,m}(\mathbb{K})$.*

Example 3.2.2. *Let $A = \begin{pmatrix} -4 & 3 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \end{pmatrix}$.*

Then

$$\begin{aligned} A \times B &= \begin{pmatrix} -4 & 3 \\ 1 & 1 \\ 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -4 \times 1 + 3 \times 0 & -4 \times -1 + 3 \times 2 & -4 \times 1 + 3 \times 3 \\ 1 \times 1 + 1 \times 0 & 1 \times -1 + 1 \times 2 & 1 \times 1 + 1 \times 3 \\ 2 \times 1 + 0 \times 0 & 2 \times -1 + 0 \times 2 & 2 \times 1 + 0 \times 3 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 10 & 5 \\ 1 & 1 & 4 \\ 2 & -2 & 2 \end{pmatrix}. \end{aligned}$$

Proposition 3.2.2. *(**Properties of Matrix Multiplication**) Let A, B and C are three matrices, then*

1. *In general $AB \neq BA$.*
2. *$(AB)C = A(BC)$.*
3. *$(A + B)C = AC + BC$, and $A(B + C) = AB + AC$.*
4. *$AB = AC$ does not imply that $B = C$.*
5. *$AB = 0$ does not imply that $A = 0$ or $B = 0$.*
6. *$AI = IA = A$, where I is identity matrix.*
7. *If a matrix is square, then we can calculate its powers, such that A^k for $k \in \mathbb{N}^*$*

$$A^k = \underbrace{A \times A \times \cdots \times A}_{k \text{ times.}}.$$

8. *$I^k = I$.*

Example 3.2.3. *Let $A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix}$ then*

1. *$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but A and B are not zero matrix.*

2. $BA = \begin{pmatrix} -5 & 5 \\ -5 & 5 \end{pmatrix} \neq AB.$

3. $AC = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = AB$, *but* $C \neq B$.

3.3 Transpose matrix

Definition 3.3.1. *The transpose of a matrix A of dimensions $n \times m$, is the matrix of dimensions $m \times n$ denoted by A^T and spoken as A-transpose, whose columns are, in order, the rows of A*

$$\text{if } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}.$$

Remark 3.3.1. *1. A square matrix is said to be* ***symmetric*** *if $A^T = A$.*

2. A square matrix is said antisymmetric, if and only if $A^T = -A$.

Example 3.3.1.

$$A = \begin{pmatrix} 3 & -1 & 4 \\ -1 & 0 & 1 \\ 4 & 1 & 2 \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} 3 & -1 & 4 \\ -1 & 0 & 1 \\ 4 & 1 & 2 \end{pmatrix}, \ A \text{ is symmetric matrix.}$$

Proposition 3.3.1. *(**Properties of Transpose matrix**) Let A and B are two matrices, then*

1. $\left(A^T\right)^T = A$,

2. $(A+B)^T = A^T + B^T$ *(if the dimensions of A and B are the same).*

3. $(AB)^T = B^T A^T$.

4. For $\alpha \in \mathbb{K}$, $(\alpha A)^T = \alpha A^T$.

3.4 Determinant

The determinant of an matrix A of order $n \times n$ is a number, denoted $\det(A)$, that is determined by A.

3.4.1 Determinant of Matrices of order 2×2.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The determinant of A, denoted by $\det(A)$ or $|A|$ is $\det A = ad - bc$.

Example 3.4.1.

$$A = \begin{pmatrix} 3 & 2 \\ -7 & -2 \end{pmatrix} \Rightarrow \det A = \begin{vmatrix} 3 & 2 \\ -7 & -2 \end{vmatrix} = 3 \times -2 - (-7) \times 2 = 8.$$

3.4.2 Determinant of Matrices of order 3×3.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

For a matrix of order 3×3, we choose a row or a column, for example here we choose the first row. Then

$$\begin{aligned} \det A &= \begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \mathbf{a}_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} \mathbf{a}_{11} \overbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}^{A_{11}} \\ &\quad + (-1)^{1+2} \mathbf{a}_{12} \overbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}^{A_{12}} + (-1)^{1+3} \mathbf{a}_{13} \overbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}^{A_{13}}. \end{aligned}$$

Where A_{ij} : is a determinat of matrix A after deleting the row i and the column j .

Example 3.4.2. *Let*

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

We choose the 1st row, then

$$\begin{aligned} \det A &= (-1)^{1+1} \times (1) \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} + (-1)^{1+2} \times (1-) \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} \\ &\quad + (-1)^{1+3} \times (2) \begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix} \\ &= -2. \end{aligned}$$

3.4.3 Determinant of Matrices of order $n \times n$.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

The determinant can be defined with respect to the ith row or to the jth as follows

1. The determinat of matrix A according to the ith row is.

$$\det A = (-1)^{i+1} a_{i1} A_{i1} + (-1)^{i+2} a_{i2} A_{i2} + \cdots + (-1)^{i+n} a_{in} A_{in}.$$

2. The determinat of matrix A according to the jth column is.

$$\det A = (-1)^{1+j} a_{1j} A_{1j} + (-1)^{2+j} a_{2j} A_{2j} + \cdots + (-1)^{n+j} a_{nj} A_{nj}.$$

Where A_{ij} : is a determinat of matrix A after deleting the ith row and the jth column.

Proposition 3.4.1. *(Properties of the determinants) Let $A, B \in \mathcal{M}_n(\mathbb{K})$, then*

1. *If A is a diagonal matrix, or a triangular matrix (Upper or Lower), then*

$$\det A = a_{11} \times a_{22} \times a_{33} \times \cdots \times a_{nn}.$$

2. $\det A^T = \det A$.
3. $\det(A.B) = \det A . \det B$.
4. *If a matrix A has two identical rows or two identical columns, then* $\det A = 0$.
5. *If, in a determinant, we add to a row a linear combination of the other rows, the determinant remains the same.*
6. *If, in a determinant, we add to a column a linear combination of the other columns, the determinant remains the same.*

Example 3.4.3.

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -6, \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 6 & -12 & 3 \end{vmatrix} = 6.$$

$$\begin{vmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0, \text{ because } column3 = column1 + column2.$$

3.5 The inverse of a matrix

Definition 3.5.1. *A square matrix A of order $n \times n$ is invertible if and only if* $\det A \neq 0$. *Its inverse is a matrix of order $n \times n$ denoted by A^{-1}, such that*

$$A \times A^{-1} = A^{-1} \times A = I_n.$$

Then, the matrix A^{-1} is defined by

$$A^{-1} = \frac{1}{\det A} adj\,(A)\,,$$

where $adj(A)$ is the ***adjugate matrix*** *of A.*

Definition 3.5.2. *The **adjugate matrix** of A is defined by*

$$\begin{aligned} adj(A) &= \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n1} & \cdots & C_{nn} \end{pmatrix}^T \\ & \text{and} \\ C_{ij} &= (-1)^{i+j} A_{ij}. \end{aligned}$$

Where A_{ij} : is a determinat of matrix A after deleting the row i and the column j.

Example 3.5.1. *Let the matrix*

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 2 & 3 & 1 \end{pmatrix}.$$

After a calculation, we find $\det A = -2 \neq 0$, *then A is invertiable, so*

$$\begin{aligned} & adj(A) \\ = & \begin{pmatrix} (-1)^{1+1}\begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \end{pmatrix}^T \\ = & \begin{pmatrix} -4 & 7 & 1 \\ 2 & -3 & -1 \\ 2 & -5 & -1 \end{pmatrix} \end{aligned}$$

Then

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} -4 & 7 & 1 \\ 2 & -3 & -1 \\ 2 & -5 & -1 \end{pmatrix} = \begin{pmatrix} 2 & \frac{-7}{2} & \frac{-1}{2} \\ -1 & \frac{3}{2} & \frac{1}{2} \\ -1 & \frac{5}{2} & \frac{1}{2} \end{pmatrix}.$$

Proposition 3.5.1. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$, then*

1. $(A^{-1})^T = \left(A^T\right)^{-1}$.
2. $\det A^{-1} = \det A$.
3. $(A.B)^{-1} = B^{-1}.A^{-1}$.

3.6 Matrices and linear applications

3.6.1 Matrices associated with a linear application

Let $\mathbb{E}$ and $\mathbb{F}$ be two vector spaces such that $\dim \mathbb{E} = n$ and $\dim \mathbb{F} = m$. Let $f : \mathbb{E} \to \mathbb{F}$ be a linear applicaton, and let

$$\mathcal{B}_{\mathbb{E}} = \{u_1, u_2, \cdots, u_n\}, \text{ basis of } \mathbb{E}, \text{ and } \mathcal{B}_{\mathbb{F}} = \{v_1, v_2, \cdots, v_m\}, \text{ basis of } \mathbb{F}.$$

Then

$$\begin{cases} f(u_1) = a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m \to 1^{st} \text{ column} \\ f(u_2) = a_{12}v_1 + a_{22}v_2 + \cdots + a_{m2}v_m \to 2^{nd} \text{ column} \\ \vdots \\ f(u_n) = a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m \to n^{th} \text{ column.} \end{cases}$$

Then, the matrix of f with respect to the bases $\mathcal{B}_{\mathbb{E}}$ and $\mathcal{B}_{\mathbb{F}}$ is the matrix denoted by $M(f)_{\mathcal{B}_{\mathbb{E}},\mathcal{B}_{\mathbb{F}}}$ such that the columns are the coordinates of the vectors $f(u_i)$ in the base $\mathcal{B}_{\mathbb{F}}$, as following

$$M(f)_{\mathcal{B}_{\mathbb{E}},\mathcal{B}_{\mathbb{F}}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ of order } m \times n.$$

Remark 3.6.1. *Let $\mathcal{B}_{\mathbb{E}}$ and $\mathcal{B}_{\mathbb{F}}$ two bases canonicls. Then the linear application $f : (\mathbb{E}, \mathcal{B}_{\mathbb{E}}) \to (\mathbb{F}, \mathcal{B}_{\mathbb{F}})$ is defined by*

$$(y_1, \cdots, y_m) = f(x_1, \cdots, x_n) \Leftrightarrow (y_1, \cdots, y_m) = M(f)_{\mathcal{B}_{\mathbb{E}},\mathcal{B}_{\mathbb{F}}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Example 3.6.1. *Let the application $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by*

$$f(x, y, z) = (y - 3z, 2x + y - z).$$

1. For

$$\mathcal{B} = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \textit{ is a canonical basis of } \mathbb{R}^3$$

$$\textit{and } \mathcal{B}' = \left\{ \varepsilon_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \textit{ is a canonical basis of } \mathbb{R}^2.$$

Determine the matrix $M(f)_{\mathcal{B},\mathcal{B}'}$:

$$\begin{cases} f(e_1) = (-2,1) = (-3)\,\varepsilon_1 + (1)\,\varepsilon_2 \\ f(e_2) = (-1,1) = (-2)\,\varepsilon_1 + (1)\,\varepsilon_2 \\ f(e_3) = (0,1) = (-1)\,\varepsilon_1 + (1)\,\varepsilon_2. \end{cases} \Rightarrow M(f)_{\mathcal{B},\mathcal{B}'} = \begin{pmatrix} 0 & 1 & -3 \\ 2 & 1 & -1 \end{pmatrix}.$$

2. For

$$\mathcal{B} = \left\{ u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \textit{ is a new basis of } \mathbb{R}^3,$$

$$\textit{and } \mathcal{B}' = \left\{ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \textit{ is a new basis of } \mathbb{R}^2.$$

Determine the matrix $M(f)_{\mathcal{B},\mathcal{B}'}$:

$$\begin{cases} f(u_1) = (-2,2) = (-2)\,v_1 + (2)\,v_2 \\ f(u_2) = (2,3) = (2)\,v_1 + (3)\,v_2 \\ f(u_3) = (-4,2) = (-4)\,v_1 + (2)\,v_2. \end{cases} \Rightarrow M(f)_{\mathcal{B},\mathcal{B}'} = \begin{pmatrix} -2 & 2 & -4 \\ 2 & 3 & 2 \end{pmatrix}.$$

3.6.2 Change of bases and transition matrix

Let $\mathbb{E}$ be a vector space such that $\dim \mathbb{E} = n$, and let

$$\mathcal{B} = \mathcal{B}_{old} = \{u_1, u_2, \cdots, u_n\}, \text{ and } \mathcal{B}' = \mathcal{B}_{new} = \{u'_1, u'_2, \cdots, u'_n\}, \text{ two bases of } \mathbb{E}.$$

Then, the **transition matrix** from $\mathcal{B}_{old}$ to $\mathcal{B}_{new}$ is the square matrix P such that the columns are the coordinates of the vectors of the new base $\mathcal{B}'$ in the old base $\mathcal{B}$, as following

$$\begin{cases} u'_1 = a_{11}u_1 + a_{21}u_2 + \cdots + a_{n1}u_n \to 1^{st} \text{ column} \\ u'_2 = a_{12}u_1 + a_{22}u_2 + \cdots + a_{n2}u_n \to 2^{nd} \text{ column} \\ \vdots \\ u'_n = a_{1n}u_1 + a_{2n}\varepsilon u_2 + \cdots + a_{nn}u_n \to n^{th} \text{ column.} \end{cases}$$

then

$$P_{\mathcal{B}_{old} \to \mathcal{B}_{new}} = P_{\mathcal{B} \to \mathcal{B}'} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \text{ of order } n \times n.$$

and we write

$$(\mathbb{E}, \mathcal{B}_{new}) \xrightarrow{P} (\mathbb{E}, \mathcal{B}_{old}).$$

Example 3.6.2. *Determine the transition matrix from $\mathcal{B}_{old}$ to $\mathcal{B}_{new}$ of $\mathbb{R}^2$, such that*

1. *For*

$$\begin{aligned}\mathcal{B}=\mathcal{B}_{old} &= \left\{e_1=\begin{pmatrix}1\\0\end{pmatrix}, e_2=\begin{pmatrix}0\\1\end{pmatrix}\right\},\\ \text{and } \mathcal{B}' &= \mathcal{B}_{new}=\left\{u_1'=\begin{pmatrix}6\\7\end{pmatrix}, u_2'=\begin{pmatrix}-4\\5\end{pmatrix}\right\}.\end{aligned}$$

Then

$$\begin{cases} u_1'=(6,7)=a_{11}e_1+a_{21}e_2=(a_{11}+a_{21},a_{21})\\ u_2'=(-4,5)=a_{12}e_1+a_{22}e_2=(a_{12}+a_{22},a_{22})\end{cases}\Rightarrow\begin{cases} a_{11}=6\wedge a_{11}+a_{21}=7\\ a_{12}=-4\wedge a_{12}+a_{22}=5\end{cases}$$

$$\Rightarrow\quad P=\begin{pmatrix}6&-4\\7&5\end{pmatrix}.$$

2. *For*

$$\begin{aligned}\mathcal{B}=\mathcal{B}_{old} &= \left\{u_1=\begin{pmatrix}1\\0\end{pmatrix}, u_2=\begin{pmatrix}1\\1\end{pmatrix}\right\},\\ \text{and } \mathcal{B}' &= \mathcal{B}_{new}=\left\{u_1'=\begin{pmatrix}-1\\2\end{pmatrix}, u_2'=\begin{pmatrix}2\\1\end{pmatrix}\right\}.\end{aligned}$$

Then

$$\begin{cases} u_1'=(-1,2)=a_{11}u_1+a_{21}u_2=(a_{11}+a_{21},a_{21})\\ u_2'=(2,1)=a_{12}u_1+a_{22}u_2=(a_{12}+a_{22},a_{22})\end{cases}$$

$$\Rightarrow\quad\begin{cases}(-1,2)=(a_{11}+a_{21},a_{21})\\(2,1)=(a_{12}+a_{22},a_{22})\end{cases}\Rightarrow\begin{cases}a_{21}=2\wedge a_{11}+a_{21}=-1\\a_{22}=1\wedge a_{12}+a_{22}=2\end{cases}$$

$$\Rightarrow\quad P=\begin{pmatrix}-3&1\\2&1\end{pmatrix}.$$

3.6.3 Change of bases for applications

Let $\mathbb{E}$ and $\mathbb{F}$ be two vector spaces such that $\dim\mathbb{E}=n$ and $\dim\mathbb{F}=m$. Let f be a linear applicaton, and let

$$\begin{aligned}\mathcal{B}_{\mathbb{E}} &= \{u_1,u_2,\cdots,u_n\},\ \mathcal{B}_{\mathbb{E}}=\{u_1',u_2',\cdots,u_n'\},\ \text{two bases of } \mathbb{E}.\\ \mathcal{B}_{\mathbb{F}} &= \{v_1,v_2,\cdots,v_m\},\ ,\ \mathcal{B}_{\mathbb{F}}=\{v_1',v_2',\cdots,v_m'\},\ \text{two bases of } \mathbb{F}.\end{aligned}$$

Such that P is transition matrix from $\mathcal{B}_{\mathbb{E}}$ to $\mathcal{B}'_{\mathbb{E}}$ of $\mathbb{E}$ and Q is transition matrix from $\mathcal{B}_{\mathbb{F}}$ to $\mathcal{B}'_{\mathbb{F}}$ of $\mathbb{F}$

$$(\mathbb{E},\mathcal{B}'_{\mathbb{E}})\xrightarrow{P}(\mathbb{E},\mathcal{B}_{\mathbb{E}})\text{ . and }(\mathbb{F},\mathcal{B}'_{\mathbb{F}})\xrightarrow{Q}(\mathbb{F},\mathcal{B}_{\mathbb{F}})$$

And we note by $M(f)_{\mathcal{B}_\mathbb{E},\mathcal{B}_\mathbb{F}}$ the matrix of f with respect to the bases $\mathcal{B}_\mathbb{E}$ and $\mathcal{B}_\mathbb{F}$, and by $M'(f)_{\mathcal{B}'_\mathbb{E},\mathcal{B}'_\mathbb{F}}$ the matrix of f with respect to the bases $\mathcal{B}'_\mathbb{E}$ and $\mathcal{B}'_\mathbb{F}$

$$(\mathbb{E},\mathcal{B}'_\mathbb{E}) \xrightarrow{P} (\mathbb{E},\mathcal{B}_\mathbb{E})\,.\text{ and }\ (\mathbb{F},\mathcal{B}'_\mathbb{F}) \xrightarrow{Q} (\mathbb{F},\mathcal{B}_\mathbb{F})$$

Then the relationship between the matrices M and M' is

$$M' = Q^{-1}MP.$$

$$\begin{array}{ccc} (\mathbb{E},\mathcal{B}_\mathbb{E}) & \xrightarrow{M} & (\mathbb{F},\mathcal{B}_\mathbb{F}) \\ P \uparrow & & \downarrow\ Q^{-1} \\ (\mathbb{E},\mathcal{B}'_\mathbb{E}) & \xrightarrow[M']{} & (\mathbb{F},\mathcal{B}'_\mathbb{F}) \end{array}$$

or

$$(\mathbb{E},\mathcal{B}'_\mathbb{E}) \xrightarrow[P]{} (\mathbb{E},\mathcal{B}_\mathbb{E}) \overbrace{\xrightarrow[M]{}}^{M'} (\mathbb{F},\mathcal{B}_\mathbb{F}) \xrightarrow[Q^{-1}]{} (\mathbb{F},\mathcal{B}'_\mathbb{F})\,.$$

3.7 Exercises set

Exercise 3.7.1. *:*
We consider the following matrices :

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 3 \\ 2 & -2 \end{pmatrix},\ B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix},\ C = \begin{pmatrix} -2 & 2 \\ 1 & 4 \\ 3 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix},\ E = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

1. *Calculate all possible sums of two of these matrices.*
2. *Calculate $3A + 2C$ and $-2C - A$.*
3. *Find α such that $A - \alpha C$ is the zero matrix.*
4. *Find the transpose of each of the matrices above.*
5. *Check that : ${}^t(BC) = {}^t C\,{}^t B$.*
6. *State what are the feasible products of two matrices. Perform those that are possible.*

Exercise 3.7.2. *:*
We consider both the following matrices :

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},\quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

- Calculate AB, BA, $(A-B)^2$ and $A^2 - 2AB + B^2$. What can we conclude ?

Exercise 3.7.3. *:*
We consider the following matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

1. *Calculate A^2 and A^3.*
2. *Calculate $A^3 - A^2 + A - I_3$.*
3. *Express A^{-1} in terms of A^2, A and I_3.*

Exercise 3.7.4. *:*
Let a and b be two given real numbers, we consider the matrix :

$$A(a,b) = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}.$$

1. *Write the matrix $A(a,b)$ in terms of $A(1,1)$ and $A(1,0)$.*
2. *Calculate $(A(1,1))^2$, deduce that $(A(1,1))^n$, $n \in \mathbb{N}$.*
3. *Calculate $(A(a,b))^n$ based on $A(1,1)$ and $A(1,0)$, $n \in \mathbb{N}$.*
   Indication : $(x+y)^n = \sum_{k=0}^{n} C_n^k x^k y^{n-k}$.

Exercise 3.7.5. *:*
Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the linear application whose matrix in the canonical basis of $\mathbb{R}^4$ and $\mathbb{R}^3$ is

$$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & -2 & 5 & -11 \end{pmatrix}.$$

1. *Determine a kernel basis of f.*
2. *Determine a basis of the image of f. What is the rank of A ?*

Exercise 3.7.6. *:*
Determine the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Exercise 3.7.7. *:*
Calculate the determinants of the following matrices, and their inverse matrices if they exist (using simplifications if possible)

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 4 & 1 \\ -1 & 5 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & -1 \\ 6 & -1 & 1 \\ 2 & -2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & a & a^2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 3.7.8. *:*
Let g the application of $\mathbb{R}^3$ whose matrix in the canonical basis $\mathbb{B} = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ is :

$$\mathcal{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$

1. *Show that g is linear and determine g^{-1}.*
2. *Determine a basis $\mathbb{B}' = \{u_1, u_2, u_3\}$ of $\mathbb{R}^3$ such that $g(u_1) = u_1, g(u_2) = u_1 + u_2$ and $g(u_3) = u_2 + u_3$.*
3. *Determine the transition matrix $\mathcal{P}$ from $\mathbb{B}$ to $\mathbb{B}'$ as well as $\mathcal{P}^{-1}$.*
4. *Determine the matrix $\mathcal{M}'$ associated of g in the basis $\mathbb{B}'$.*
5. *Deduce $g^n(e_1)$, $g^n(e_2)$ and $g^n(e_3)$ for n is a relative integer.*

Exercise 3.7.9. *:*
Let $f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$ a linear application defined for all $P \in \mathbb{R}_2[X]$ by

$$f(P) = (1 + X)P'.$$

Lat $\beta = \{1, X, X^2\}$ the canonical basis of $\mathbb{R}_2[X]$.

1. *Determine the matrix A of f in the canonical basis β.*
2. *Show that $\beta' = \{1, 1 + X, (1 + X)^2\}$ is a basis of $\mathbb{R}_2[X]$.*
3. *Determine the trasition matrix from β to β'.*

4. *Determine the matrix B of f in the basis β'.*
5. *Calculer B^2, B^3 et B^k pour tout $k \in \mathbb{N}$.*
6. *Determine the rank of f..*
7. *Conclude whether the application$f$ is bijective.*

Exercise 3.7.10. *:*
Let $T : \mathbb{R}_2[X] \longrightarrow \mathbb{R}[X]$ an application defined for all $P \in \mathbb{R}_2[X]$ by

$$T(P) = 2P - (X-1)P'.$$

Lat $\beta = \{1, X, X^2\}$ the canonical basis of $\mathbb{R}_2[X]$.

1. *Show that T is linear.*
2. *Determine the matrix A of T in β.*
3. *Determine a polynomial P_1 such that $T(P_1) = P_1$.*
4. *Determine the kernel of T. We will denote by P_2 a direction vector of the kernel.*
5. *Show that $\beta' = \{1, P_1, P_2\}$ is a basis of $\mathbb{R}_2[X]$.*
6. *Determine the trasition matrix from β to β'.*
7. *Determine the matrix D of T in β'.*

Chapitre 4

Systems of linear equations

In this chapter, we will study systems of linear equations. This is based on the Cramer method and the matrix inversion method for the resolution. Throughout this chapter, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

4.1 Interpretations

For $n, m \in \mathbb{N}$, a system of n linear equations and m unknowns is a list of and n linear equations. It is of the form :

$$(S) : \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n \end{cases}.$$

The numbers a_{ij} are the coefficients of the system.
We can write (S) in matrix form as follows $AX = b$, such that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

We call $A = (a_{ij}) \in M_{n,m}(\mathbb{K})$ the matrix of coefficients of the system.
$b \in M_{n,1}(\mathbb{K})$ is the right-hand vector.
The vector $X \in M_{m,1}(\mathbb{K})$ is a solution of the system if and only if $AX = b$.

Example 4.1.1.

$$(S) : \begin{cases} x + 2y = 1 \\ 3x - y = 4 \\ 2x - 3y = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}.$$

In the following, we will only be interested in solving systems where the number of equations is equal to the number of unknowns. To do this, we use Cramer's method. This method is named after Gabriel Cramer, Swiss mathematician $(1704 - 1752)$.

4.2 Cramer systems

A Cramer system is a system of linear equations where the number of equations is equal to the number of unknowns and the matrix of coefficients has a non-zero determinant ($\det A \neq 0$). When the determinant is zero, the system is not Cramer.
Consider a system of linear equations (S) in the form

$$(S) \quad : \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

$$(S) \quad \Leftrightarrow \quad \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{b}.$$

4.3 Solving Methods

4.3.1 Cramer method

Cramer calculates the unknowns x_k by the method as follows

$$x_k = \frac{\det A_k}{\det A},$$

where A_k : the matrix A after replacing the kth row by the vector b.

4.3.2 Inversion method

Let system of linear equations where the associated matrix is invertiable ($\det A \neq 0$), then

$$AX = b \Leftrightarrow X = A^{-1}b.$$

Example 4.3.1. *Let the system*

$$\begin{cases} x + 2y + 3z = -4 \\ w - y - z = 2 \\ 3x + y - 2z = 11 \end{cases} \Leftrightarrow \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -1 \\ 3 & 1 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix}.$$

The system (S) *admits a single solution if and only if* $\det(A) \neq 0$.
We have

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & -1 \\ 3 & 1 & -2 \end{vmatrix} = 13.$$

Since $\det(A) = 13 \neq 0$, *then* (S) *has only one solution.*
- By using Cramer method :

$$x = \frac{\begin{vmatrix} \mathbf{-4} & 2 & 3 \\ \mathbf{2} & -1 & -1 \\ \mathbf{11} & 1 & -2 \end{vmatrix}}{\det A} = \frac{13}{13} = 1,$$

$$y = \frac{\begin{vmatrix} 1 & \mathbf{-4} & 3 \\ 1 & \mathbf{2} & -1 \\ 3 & \mathbf{11} & -2 \end{vmatrix}}{\det A} = \frac{26}{13} = 2,$$

$$z = \frac{\begin{vmatrix} 1 & 2 & \mathbf{-4} \\ 1 & -1 & \mathbf{2} \\ 3 & 1 & \mathbf{11} \end{vmatrix}}{\det A} = \frac{-39}{13} = -3.$$

- By using inversion method :

Then, the solution X **of** (S) **is given by** $X = A^{-1} \cdot b$

$$A^{-1} = \tfrac{1}{13} \begin{pmatrix} 3 & 7 & 1 \\ -1 & -11 & 4 \\ 4 & 5 & -3 \end{pmatrix}.$$

Finally, $X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tfrac{1}{13} \begin{pmatrix} 3 & 7 & 1 \\ -1 & -11 & 4 \\ 4 & 5 & -3 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}.$

4.4 Exercises set

Exercise 4.4.1. *:*
Solve in $\mathbb{C}$ *the following linear systems :*

$$(C_1): \begin{cases} x + 2iy = i + 2 \\ ix - 3iy = -4 \end{cases} \quad (C_2): \begin{cases} x - iy = 2 \\ ix + 2y = -1 \end{cases}$$

Exercise 4.4.2. *:*
Solve the following systems by two methods :

$$(S_1): \begin{cases} 2x+2y+z=1 \\ 2x+y-z=2 \\ 3x+y+z=3 \end{cases} \quad (S_2): \begin{cases} x+2y+3z=0 \\ x+3y+z=0 \\ 5x+y-4z=0 \end{cases} \quad (S_3): \begin{cases} x-2y+3z=0 \\ 2x+4y-2z=0 \\ x+5y-4z=0 \end{cases}$$

Exercise 4.4.3. *:*
Let $k \in \mathbb{R}$, we consider the following matrix

$$A = \begin{pmatrix} 1 & k & -2 \\ 1 & k+1 & k-2 \\ 2 & 2k+1 & 2k-4 \end{pmatrix}$$

1. *Determine for which values of k, the matrix A is invertible.*
2. *Solve the system*

$$(S): \begin{cases} x+ky-2z=-1 \\ x+(k+1)y+(k-2)z=3 \\ 2x+(2k+1)y+(2k-4)z=-2 \end{cases}$$

Exercise 4.4.4. *:*
Solve and discuss according to the values of $\alpha \in \mathbb{R}$, the following system of linear equations :

$$(S): \begin{cases} y+\alpha z+t=0 \\ x+y+z+t=1 \\ x+2y+z+2t=1 \\ 3x+3y+4z+\alpha t=4 \end{cases}$$

Chapitre 5

Reduction of matrix

Matrix reduction is the matrix version of finite-dimensional endomorphism reduction. The matrix of such an endomorphism then depends on the base chosen to represent it, but any change of base gives a similar matrix. Reducing a matrix consists of looking for the simplest possible similar matrix : in the best case, a diagonal matrix (this is then a diagonalization), otherwise a triangular matrix (this is then a trigonalization).

5.1 Eigenvalues and Eigenvectors

Definition 5.1.1. *Let $f : \mathbb{E} \to \mathbb{E}$ be an endomorphisms (linear application), v a vector nonzero in $\mathbb{E}$ and λ a scalar in $\mathbb{K}$, if*

$$f(v) = \lambda v.$$

Then λ is an eigenvalue of f and v is an eigenvector of f.

Theorem 5.1.1. *λ is an eigenvalue of f if and anly if*

$$\ker\left(f - \lambda I_{\mathbb{E}}\right) \neq \{0_{\mathbb{E}}\}.$$

Where $I_{\mathbb{E}}$ is an identity application.

Definition 5.1.2. *Let A be a square matrix in $\mathcal{M}_n(\mathbb{R})$, v a vector nonzero in $\mathbb{R}^n$ and λ a scalar in $\mathbb{R}$, if*

$$Av = \lambda v.$$

Then λ is an eigenvalue of A and v is an igenvector of A.

Example 5.1.1.

$$\begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

then $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector to $\begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}$ and 3 is its an eigenvalue associated.

Also we have

$$\begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \\ 15 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -5 \end{pmatrix},$$

then the eigenvectors associated with the eigenvalue -3 *is the vector* $v = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$.

5.2 Characteristic Polynomial

Definition 5.2.1. *Let* A *be a square matrix in* $\mathcal{M}_n(\mathbb{R})$. *We call the characteristic polynomial of* A *the polynomial denoted* P_A *of degree* n, *defined by*

$$P_A(\lambda) = \det(A - \lambda I_n).$$

Proposition 5.2.1. *"**Eigenvalues**" The eigenvalues of* A *are the roots of the characteristic polynomial* $P_A(\lambda)$.

Theorem 5.2.2. *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *be an invertible matrix. The following are true :*

1. *If* A *is triangular, then the diagonal elements of* A *are the eigenvalues of* A.
2. *If* λ *is an eigenvalue of* A *with eigenvector* v, *then* $\frac{1}{\lambda}$ *is an eigenvalue of* A^{-1} *with eigenvector* v.
3. *If* λ *is an eigenvalue of* A *then* λ *is an eigenvalue of* A^T.
4. *The sum of the eigenvalues of* A *is equal to* $tr(A)$, *the trace of* A.
5. *The product of the eigenvalues of* A *is the equal to* $\det(A)$, *the determinant of* A.
6. $\det(A) = P_A(0)$.

Theorem 5.2.3. *"**Invertible Matrix Theorem**" Let* $A \in \mathcal{M}_n(\mathbb{R})$ *be an matrix. The following statements are equivalent.*

(a) A *is invertible.*

(b) A *does not have an eigenvalue of* 0.

Definition 5.2.2. *"**Eigen-subspace or Eigenspace** " Let* A *be a square matrix in* $\mathcal{M}_n(\mathbb{R})$. *let* λ *be the eigenvalue of* A. *We call the eigenspace associated with* λ *the vector subspace* E_λ *defined by*

$$E_\lambda = \{v \in \mathbb{R}^n : Av = \lambda v\}.$$

Example 5.2.1. *Let the matrix* A *be defined by*

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

to find the characteristic polynomial of A, *we calculate* $\det(A - \lambda I_2)$

$$\det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4).$$

Then, the eigenvalues of A are the roots of $P_A(\lambda) = 0$ when $\lambda = 1$ and $\lambda = 4$.

The eigenspaces associated with** 1 **and** 4 **:

(a) For $\lambda = 1$:

$$E_1 = \left\{ v \in \mathbb{R}^2 : Av = v \right\},$$

then let $v = (x, y) \in \mathbb{R}^2$

$$\begin{aligned} Av &= v \Leftrightarrow \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \\ &\Leftrightarrow \begin{cases} 3x + 2y = x \\ x + 2y = y \end{cases} \Rightarrow x = -y \end{aligned}$$

hence

$$E_1 = \left\{ (x, y) \in \mathbb{R}^2 : x = -y \right\} = \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle.$$

Let $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ be a proper vector associated with $\lambda = 1$.

(b) For $\lambda = 4$: We have

$$E_4 = \left\{ v \in \mathbb{R}^2 : Av = 4v \right\},$$

then let $v = (x, y) \in \mathbb{R}^2$

$$\begin{aligned} Av &= 4v \Leftrightarrow \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix} \\ &\Leftrightarrow \begin{cases} 3x + 2y = 4x \\ x + 2y = 4y \end{cases} \Rightarrow x = 2y, \end{aligned}$$

hence

$$E_4 = \left\{ (x, y) \in \mathbb{R}^2 : x = 2y \right\} = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle.$$

Let $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ be a proper vector associated with $\lambda = 4$.

5.2.1 Cayley-Hamilton Theorem

In this part, we will study the annulator polynomials of a matrix $A \in \mathcal{M}_n(\mathbb{R})$. Consider a non-zero polynomial $P \in \mathbb{R}_n[X]$

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0.$$

The matrix $P(A)$ is given by the following expression

$$P(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A^1 + a_0 A^0.$$

where, $A^k = \underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$, and $A^0 = I$.

Otherwise if P is of the form

$$P(X) = (X - \lambda_1)^{k_1}(X - \lambda_2)^{k_2} \cdots (X - \lambda_p)^{k_p}$$

then,

$$P(A) = (A - \lambda_1 I)^{k_1}(A - \lambda_2 I)^{k_2} \cdots (A - \lambda_p I)^{k_p}.$$

Definition 5.2.3. *"**Annulator polynomail**" Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. We say that a non-zero polynomial $P \in \mathbb{R}_n[X]$ annulator of A if and only if $P(A) = 0$ (zero matrix).*

Example 5.2.2. *Let the matrix*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The polynomail $P(X) = X^2 - 4X + 3$ is an annulator polynomail of A.

$$\begin{aligned} P(A) &= A^2 - 4A + 3I_2 \\ &= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - 4\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Theorem 5.2.4. *"**Cayley-Hamilton Theorem**" Let $A \in \mathcal{M}_n(\mathbb{R})$ be a square matrix. The characteristic polynomial $P_A(X)$ is annulator polynomail of A, that is to say*

$$P_A(A) = 0$$

Application the annulator polynomail :

Let $P(X) = X^2 - 4X + 3$ is annulator polynomail of matrix $A \in \mathcal{M}_2(\mathbb{R})$. So acorrding to the theorem 5.2.3 the matrix A is invertible $(P(0) = \det A = 3)$, on the other hand we have

$$\begin{aligned} P(A) &= A^2 - 4A + 3I_2 = 0 \Rightarrow A^2 - 4A = -3I_2 \\ &\Rightarrow A(A - 4I_2) = -3I_2 \\ &\Rightarrow A^{-1} = \frac{1}{-3}(A - 4I_2) = \frac{1}{-3}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}. \end{aligned}$$

5.3 Reduction of matrix

5.3.1 Diagonalization

Consider a matrix A of $\mathcal{M}_n(\mathbb{R})$, we can assume that A is the matrix representation of a linear application $T : \mathbb{R}^n \to \mathbb{R}^n$ in a basis of $\mathbb{R}^n$. In this part, we will look for a base of $\mathbb{R}^n$ such that A becomes diagonal in it. In this case, we say that A is diagonalizable. We will see here that this basis is that of the eigenvectors of A.

Definition 5.3.1. *(Diagonalizable matrix) Let A be a square matrix in $\mathcal{M}_n(\mathbb{R})$. We say that A is diagonalizable if and only if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$ and a diagonal matrix $D \in \mathcal{M}_n(\mathbb{R})$ such that*

$$D = P^{-1}AP \Leftrightarrow A = PDP^{-1}.$$

Theorem 5.3.1. *Let A be a square matrix in $\mathcal{M}_n(\mathbb{R})$ such that $\lambda_1, \lambda_2, \cdots, \lambda_p$ are its eigenvalues. We say that A is diagonalizable if and only if*

$$\dim \mathbb{R}^n = n = \dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots + \dim E_{\lambda_p}.$$

Example 5.3.1. *Diagonalization the following matrix*

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

1. ***Step** 1 : Calculation of eigenvalues. The characteristic polynomial of A is given by*

$$P_\lambda(A) = \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-2)^2.$$

   The eigenvalues of A are $\{-1, 2\}$

2. ***Step** 2 : Determine the eigenspaces associated with a eigenvalues of A.*

   b1) For $\lambda_1 = -1$.

$$E_{\lambda_1} = \left\{v \in \mathbb{R}^3 : Av = -v\right\}, v = (x, y, z)$$

   so

$$Av = -v \Leftrightarrow \begin{cases} x + y - z = -x \\ x + y + z = -y \\ -x + y + z = -z \end{cases} \Leftrightarrow \begin{cases} x = -y \\ z = -y \end{cases},$$

   then the eigenspaces for $\lambda_1 = -1$ is $E_{\lambda_1} = \left\langle \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\rangle$, $\dim E_{\lambda_1} = 1$.

   b2) For $\lambda_2 = 2$.

$$E_{\lambda_2} = \left\{v \in \mathbb{R}^3 : Av = 2v\right\}, v = (x, y, z)$$

so

$$Av = 2v \Leftrightarrow \begin{cases} x+y-z=2x \\ x+y+z=2y \\ -x+y+z=2z \end{cases} \Leftrightarrow \begin{cases} x=y-z \\ x=y-z \end{cases},$$

then the eigenspaces associated with $\lambda_2 = -2$ *is*

$$E_{\lambda_2} = \{(y-z, y, z), y, z \in \mathbb{R}\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle, \dim E_{\lambda_2} = 2.$$

3. ***Step*** *3 : Matrix presentation. According to the diagonalization theorem we have* $\dim E_{\lambda_1} + \dim E_{\lambda_2} = 1 + 2 = \dim \mathbb{R}^3$, *then* A *is diagonalizable. The invertible matrix* P *(**Transition matrix**) is that of the eigenvectors and the diagonal matrix* D *contains the eigenvalues, so*

$$P = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \text{ and } A = PDP^{-1}.$$

5.3.2 Trigonalization

Definition 5.3.2. *(Trigonalizable matrix) Let* A *be a square matrix in* $\mathcal{M}_n(\mathbb{R})$. *We say that* A *is trigonalizable if and only if there exists an invertible matrix* $Q \in \mathcal{M}_n(\mathbb{R})$ *and a triangular matrix* $T \in \mathcal{M}_n(\mathbb{R})$ *such that*

$$T = Q^{-1}AQ \Leftrightarrow A = QTQ^{-1}.$$

Theorem 5.3.2. *(Trigonalization) Let* A *be a square matrix in* $\mathcal{M}_n(\mathbb{R})$ *such that* $\lambda_1, \lambda_2, \cdots, \lambda_p$ *are its eigenvalues. We say that* A *is trigonalizable if and only if the characteristic polynomial* $P_A(\lambda)$ *has only real roots and*

$$\dim \mathbb{R}^n = n \neq \dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots + \dim E_{\lambda_p}.$$

Example 5.3.2. *Trigonalization the following matrix*

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

a) ***Step*** *1 : Calculation of eigenvalues. The characteristic polynomial of* A *is given by*

$$P_\lambda(A) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2.$$

then the eigenvalues of A *are* $\{1, 2\}$, *(1 simple root, 2 double root)*

b) ***Step*** *2 : Determine the eigenspaces associated with a eigenvalues of* A :

b1) For $\lambda_1 = 1$.

$$E_{\lambda_1} = \left\{ v \in \mathbb{R}^3 : Av = v \right\}, v = (x, y, z)$$

so

$$Av = v \Leftrightarrow \left\{ \begin{array}{c} 2x + z = x \\ 2y + z = y \\ x - y + z = z \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} x = y \\ z = -y \end{array} \right. ,$$

then the eigenspaces for $\lambda_1 = 1$ *is*

$E_{\lambda_1} = \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle$, $\dim E_{\lambda_1} = 2$.

b2) For $\lambda_2 = 2$.

$$E_{\lambda_2} = \left\{ v = (x, y, z) \in \mathbb{R}^3 : Av = 2v \right\},$$

so

$$Av = 2v \Leftrightarrow \left\{ \begin{array}{c} 2x + z = 2x \\ 2y + z = 2y \\ x - y + z = 2z \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} x = y \\ z = 0 \end{array} \right. ,$$

then the eigenspaces for $\lambda_2 = 1$ *is*

$E_{\lambda_2} = \{(y, y, 0), y \in \mathbb{R}\} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$, $\dim E_{\lambda_2} = 1$.

c) ***Step*** *3 : Matrix presentation. According to the trigonalization theorem we have* $\dim E_{\lambda_1} + \dim E_{\lambda_2} = 1 + 1 \neq \dim \mathbb{R}^3$, *then* A *is trigonalizable.*
For trigonalizable the matrix B *it is enough to complete the basis* $\left\{ v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ *for* $\mathbb{R}^3$. *We choose for example* $y = z = 0$), *then* $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, *so the transition matrix is*

$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, *and* $T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ *and*

$$A = QTQ^{-1}.$$

5.3.3 Some applications

5.3.3.1 Powers of matrix

Let A be a square matrix in $\mathcal{M}_n(\mathbb{R})$ and let A is diagonalizable, so if and only there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{R})$ and a diagonal matrix $D \in \mathcal{M}_n(\mathbb{R})$ such that

$$D = P^{-1}AP \Leftrightarrow A = PDP^{-1}.$$

Then

$$A^k = PD^kP^{-1}, \forall k \in \mathbb{N}.$$

Where

$$D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix}, \text{ such that the } \lambda_1, \lambda_2, ...\lambda_n \text{ are the eigenvalues of } A.$$

5.3.3.2 Solving a system of linear differential equations

Example 5.3.3. *:*
See the correction of the exercise 3 of the synthesis exam 2023.

5.4 Exercises set

Exercise 5.4.1. *:*
Let f be an endomorphism whose matrix in the canonical basis of $\mathbb{R}^3$ is

$$\mathcal{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

1. *Find the eigenvalues and eigensubspaces of f.*
2. *Diagonalize f if possible.*

Exercise 5.4.2. *:*
We consider the following matrix

$$A = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

1. *Calculate the characteristic polynomial of A.*
2. *Determine the eigenvalues and the eigenvectors of A.*

3. *Is A diagonalizable ?*
4. *Calculate A^m for every element m of $\mathbb{N}^*$.*

Exercise 5.4.3. *:*
Diagonalize or trigonalize the following matrices :

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}, D = \begin{pmatrix} -3 & -3 & 2 \\ 1 & 1 & -2 \\ 2 & 4 & -4 \end{pmatrix}.$$

Exercise 5.4.4. *:*

1. *Diagonalize the matrix*
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 4 & 2 & 0 \end{pmatrix}.$$
2. *Solve the following system of differential equations :*
$$\begin{cases} x'(t) = x(t) + 2y(t) + 3z(t) \\ y'(t) = 2x(t) + y(t) + 3z(t) \\ z'(t) = 4x(t) + 2y(t) \end{cases}$$

Exercise 5.4.5. *:*

1. *Diagonalize the matrix*
$$A = \begin{pmatrix} 7 & -3 & -4 \\ -4 & 6 & 4 \\ 5 & -3 & -2 \end{pmatrix}.$$
2. *Deduce that A^n.*
3. *We consider the sequences (u_n) ; (v_n) ; (w_n) defined by :*
$$\begin{cases} u_{n+1} = 7u_n - 3v_n - 4w_n \\ v_{n+1} = -4u_n + 6v_n + 4w_n \\ w_n = 5u_n - 3v_n - 2w_n \end{cases}$$

- *Express u_n ; v_n ; w_n as a function of n.*

Chapitre 6

Some synthesis exams and its corrections

In this chapter, we will group together most of the synthesis exams with detailed standard corrections for the recent past years.

6.1 Synthesis exam 2023

Exercise 6.1.1. *:*
Answer by true or false, with justification

1. *0 is the eigenvalue of the matrix $M \iff$ The matrix M is invertible.*
2. *The matrix M is not invertible $\implies$ The matrix M is not diagonalizable.*
3. *$M, N \in \mathcal{M}_n(\mathbb{R})$, and N is invertible, then :* $\det(N^{-1}M.N) = \det M$.
4. *$M, N \in \mathcal{M}_n(\mathbb{R})$, then :* $\det(M+N) = \det M + \det N$.

Exercise 6.1.2. *:*
We define the application T by

$$T : \mathbb{R}_2[X] \to \mathbb{R}_2[X], \quad T(P) = P + 2P'.$$

Where $\mathbb{R}_2[X]$ the space of polynomials of degree $\leq$2, and P' the derivative of P.

1. *Show that T is linear.*
2. *Determine the matrix A of T in the canonical basis $\mathcal{B} = \{1, X, X^2\}$.*
3. *Let $\mathcal{B}' = \left\{1, X-3, (X+2)^2\right\}$ a family of $\mathbb{R}_2[X]$.*
   - *3.1 Show that $\mathcal{B}'$ is a basis of $\mathbb{R}_2[X]$.*
   - *3.2 Determine the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.*
   - *3.3 Find the matrix A' of T in the basis $\mathcal{B}'$.*

Exercise 6.1.3. *:*
We consider the following matrix

$$B = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix}.$$

1. *Determine an annulator polynomail of the matrix* B.
2. *Is* B *diagonalizable ? If yes, determine the invertible matrix* P *and the diagonal matrix* D *such that* $D = P^{-1}BP$.
3. *We consider on* $\mathbb{R}$ *the following differential system :* $(S_d) : \begin{cases} f' = -4f - 6g \\ g' = 3f + 5g \\ h' = 3f + 6g + 5h \end{cases}$,
   where $f : x \mapsto f(x), g : x \mapsto g(x)$, *and* $h : x \mapsto h(x)$ *are differentiable functions.*
   - *Solve the differential system* (S_d).

6.2 Correction of synthesis exam 2023

Solution 6.2.1. *of (6.1.1) :*

1. *True :* 0 *is proper value of matrix* $M \Leftrightarrow 0$ *is a root of polynomial characteristic* $P_M(\lambda)$ *of* M *i.e.* $P_M(0) = \det(M - 0I) = \det M = 0$.
2. *False :* $M = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ *not invertible but it is diagonalizable.*
3. *True :*

$$\begin{aligned} \det\left(N^{-1}MN\right) &= \det\left(N^{-1}\right)\det(M)\det(N) = \det(M).\det\left(N^{-1}\right).\det(N) \\ &= \det(M)\det\left(N^{-1}N\right) = \det(M)\det(I) = \det M. \end{aligned}$$

4. *False :* $M = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, N = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \Longrightarrow \det(M + N) = 0,$
   but $\det M + \det N = -4$.

Solution 6.2.2. *of (6.1.2) :*
1) T *is linear. Let* $P, Q \in \mathbb{R}_2[X]$ *and* $\alpha \in \mathbb{R}$, *then we have*

$$\begin{aligned} T(P+Q) &= P + Q + 2(P+Q)' = P + 2P' + (Q + 2Q') \\ &= T(P) + \alpha T(Q). \end{aligned}$$

Moreover,

$$\begin{aligned} T(\alpha P) &= \alpha P + 2(\alpha P)' = \alpha(P + 2P') \\ &= \alpha T(P). \end{aligned}$$

Or

$$\begin{aligned} T(P + \alpha Q) &= P + \alpha Q + 2(P + \alpha Q)' = P + 2P' + \alpha(Q + 2Q') \\ &= T(P) + \alpha T(Q). \end{aligned}$$

2) *Let us determine the matrix A of T in the canonical basis $\mathcal{B} = \{1, X, X^2\}$.*

$$\begin{cases} T(1) = 1 = 1 + 0.X + 0.X^2. \\ T(X) = X + 1 = 1 + 1.X + 0.X^2. \\ T(X^2) = X^2 + 2X = 0 + 2.X + 1.X^2 \end{cases}$$

Then, the matrix A of T in the canonical basis $\mathcal{B} = \{1, X, X^2\}$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

3) *Let $\mathcal{B}' = \left\{1, X-3, (X+2)^2\right\}$ is a base of $\mathbb{R}_2[X]$*
3.1) *Let us determine the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$:*

$$\begin{cases} 1 = 1 + 0.X + 0.X^2. \\ X - 3 = -3 + 1.X + 0.X^2. \\ (X+2)^2 = 4 + 4.X + 1.X^2. \end{cases}$$

Thus, the transition matrix Q is given by

$$Q = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.2) *Let us find the matrix A' of T in the basis $\mathcal{B}'$ by using the matrices A and Q, we arrive at*

$$\left(\mathbb{R}_2[X], \mathcal{B}'\right) \xrightarrow{Q} \left(\mathbb{R}_2[X], \mathcal{B}\right) \xrightarrow{A} \left(\mathbb{R}_2[X], \mathcal{B}\right) \xrightarrow{Q^{-1}} \left(\mathbb{R}_2[X], \mathcal{B}'\right),$$

so

$$A' = Q^{-1}AQ.$$

Calculate the matrix Q^{-1}

$$\begin{aligned} \det Q &= 1, \text{ and } com(Q) = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -16 & -4 & 1 \end{pmatrix}, \\ \text{then } Q^{-1} &= \frac{1}{\det Q}{}^{t}(comQ) = \begin{pmatrix} 1 & 3 & -16 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

So

$$A' = Q^{-1}.A.Q = \begin{pmatrix} 1 & 3 & -16 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution 6.2.3. *of (6.1.3) :*

1) *The characteristic polynomial of the matrix B is an annulator polynomial (according to the Cayley-Hamilton theorem), then*

$$P_B(\lambda) = \det(B - \lambda I_3) = (5 - \lambda)(\lambda + 1)(\lambda - 2).$$

3) *Is B diagonalizable ? If yes, determine the invertible matrix P and the diagonal matrix D such that $D = P^{-1}BP$.*

3.1) *The eigenvalues are the roots of $P_B(\lambda)$ i.e. $\lambda_1 = 5, \lambda_2 = -1$, and $\lambda_3 = 2$.*

3.2) *Let us determine the eigensubspaces associated with the eigenvalues :*

a) *Eigensubspace associated with $\lambda_1 = 5$, $E_{\lambda_1} = \{v \in \mathbb{R}^3 : Bv = 5v\}$, then*

$$Bv = 5v \Longrightarrow \begin{cases} -4x - 6y = 5x \\ 3x + 5y = 5y \\ 3x + 6y + 5z = 5z \end{cases} \Longrightarrow \begin{cases} x = 0 \\ y = 0 \\ z \in \mathbb{R} \end{cases}.$$

Therefore, the eigensubspace is given by

$$E_{\lambda_1} = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, z \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

b) *Eigensubspace associated with $\lambda_2 = -1$, $E_{\lambda_2} = \{v \in \mathbb{R}^3 : Bv = -v\}$, then*

$$Bv = -v \Longrightarrow \begin{cases} -4x - 6y = -x \\ 3x + 5y = -y \\ 3x + 6y + 5z = -z \end{cases} \Longrightarrow \begin{cases} x = -2y \\ z = 0 \end{cases}.$$

So, the eigensubspace associated is given by

$$E_{\lambda_2} = \left\{ \begin{pmatrix} -2y \\ y \\ 0 \end{pmatrix}, y \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\rangle.$$

c) *Eigensubspace associated with $\lambda_3 = 2$, $E_{\lambda_3} = \{v \in \mathbb{R}^3 : Bv = 2v\}$,*

then

$$Bv = -v \Longrightarrow \left\{ \begin{array}{c} -4x - 6y = 2x \\ 3x + 5y = 2y \\ 3x + 6y + 5z = 2z \end{array} \right. \Longrightarrow \left\{ \begin{array}{c} y = -x \\ z = x \end{array} \right. .$$

So, the eigensubspace associated is given by

$$E_{\lambda_3} = \left\{ \begin{pmatrix} x \\ -x \\ x \end{pmatrix}, x \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle .$$

Hence,

$$\dim \mathbb{R}^3 = 3 = \dim E_{\lambda_1} + \dim E_{\lambda_2} + \dim E_{\lambda_3},$$

we conclude that B is diagonalizable.

3.3) *Matrix presentation : The transition matrix P and the diagonal matrix D*

$$\begin{aligned} P &= \begin{pmatrix} 0 & -2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}. \\ D &= \begin{pmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \end{aligned}$$

4) *The resolution of the differential system (S_d) :*

$$\begin{aligned} (S_d) \quad &\Leftrightarrow \quad \begin{pmatrix} f' \\ g' \\ h' \end{pmatrix} = P.D.P^{-1} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \\ &\Leftrightarrow \quad P^{-1} \begin{pmatrix} f' \\ g' \\ h' \end{pmatrix} = D.P^{-1} \begin{pmatrix} f \\ g \\ h \end{pmatrix}. \end{aligned}$$

We make the following change of variables, we put

$$\begin{pmatrix} F \\ G \\ H \end{pmatrix} = P^{-1} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \Longrightarrow \begin{pmatrix} F' \\ G' \\ H' \end{pmatrix} = D \begin{pmatrix} F \\ G \\ H \end{pmatrix}.$$

We apply differential calculus, then

$$\left\{\begin{array}{l} F'=5F \\ G'=-G \\ H'=2H \end{array}\right. \Rightarrow \left\{\begin{array}{l} F(t)=C_1e^{5t} \\ G(t)=C_2e^{-t} \\ H(t)=C_3e^{2t} \end{array}\right. \Rightarrow \left\{\left(\begin{array}{c} f(t) \\ g(t) \\ h(t) \end{array}\right)=P\left(\begin{array}{c} C_1e^{5t} \\ C_2e^{-t} \\ C_3e^{2t} \end{array}\right)\right.$$

$$\Rightarrow \left\{\begin{array}{l} f(t)=-2C_2e^{-t}+C_3e^{2t} \\ g(t)=C_2e^{-t}-C_3e^{2t} \\ h(t)=C_1e^{-t}+C_3e^{2t} \end{array}\right. .$$

6.3 Interrogation exam 2023

Exercise 6.3.1. *:*
Answer only one question, (I) or (II) :

(I). Can we determine x, y in $\mathbb{R}$ such that the vector $V=(-2,x,y,3)$ belongs to the vector subspace of $\mathbb{R}^4$ generated by $e_1=(1,-1,1,2)$ and $e_2=(-1,2,3,1)$.

(II). In $\mathbb{R}^3$, we consider the following two vector subspaces :

$$\begin{array}{rcl} E_1 & = & \left\{(x,y,z)\in\mathbb{R}^3; x+ay+z=0,\ a\in\mathbb{R}\right\} \\ & & et \\ E_2 & = & \left\{(x,y,z)\in\mathbb{R}^3; x=-y=z\right\}. \end{array}$$

For what values of a we have : $\mathbb{R}^3=E_1\oplus E_2$.

Exercise 6.3.2. *:*
Let $f:\mathbb{R}^2\to\mathbb{R}^3$ be a linear application whose matrix is in the canonical bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ is :

$$A=\left(\begin{array}{cc} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{array}\right)$$

1. *Calculate $Im(f)$ and $\ker(f)$.*
2. *f is it bijective ?*
3. *What is the rank of A ?*

Exercise 6.3.3. *:*
Let $b\in\mathbb{R}^*$. We consider the matrix

$$A(b)=\left(\begin{array}{cc} 1+b & -b \\ b & 1-b \end{array}\right).$$

1. *Determine the matrix M such that $A(b)=I_2+bM$.*
2. *Calculate M^2, deduce $(A(b))^{100}$.*

6.4 Correction of interrogation exam 2023

Solution 6.4.1. *of (6.3.1) :*
I). Can we determine x, y in $\mathbb{R}$ such that the vector $V = (-2, x, y, 3)$ belongs to the vector subspace of $mathbbR^4$ generated by $e_1 = (1, -1, 1, 2)$ and $e_2 = (-1, 2, 3, 1)$.

We have $V \in \langle e_1, e_2 \rangle \Longrightarrow \exists! \alpha, \beta \in \mathbb{R}$ such that $V = \alpha e_1 + \beta e_2$.
Let $\alpha, \beta \in \mathbb{R}$, then

$$V = \alpha e_1 + \beta e_2 \Longrightarrow \begin{cases} -2 = \alpha - \beta \\ x = -\alpha + 2\beta \\ y = \alpha + 3\beta \\ 3 = 2\alpha + \beta \end{cases}$$

$$\Longrightarrow \begin{cases} \alpha = \frac{1}{3} \\ \beta = \frac{7}{3} \\ x = \frac{13}{7} \\ y = \frac{22}{3} \end{cases}.$$

(II). In $\mathbb{R}^3$, we consider the following two vector subspaces :

$$\begin{aligned} E_1 &= \left\{(x, y, z) \in \mathbb{R}^3; x + ay + z = 0,\ a \in \mathbb{R}\right\} \\ &= \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -a \end{pmatrix} \right\rangle \Longrightarrow \dim E_1 = 2, \\ &\text{and} \\ E_2 &= \left\{(x, y, z) \in \mathbb{R}^3; x = -y = z\right\} \\ &= \left\langle \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\rangle \Longrightarrow \dim E_2 = 1. \end{aligned}$$

Let's determine the values of a such that : $\mathbb{R}^3 = E_1 \oplus E_2$.
We have $\mathbb{R}^3 = E_1 \oplus E_2 \Longleftrightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -a \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ form a basis of $\mathbb{R}^3$.

First method :

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -a & -1 \end{vmatrix} \neq 0 \Longrightarrow a \in \mathbb{R} - \{2\}.$$

Second method : *Let* $\alpha, \beta, \gamma \in \mathbb{R}$, *then*

$$\begin{aligned}
\alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -a \end{pmatrix} \gamma \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} &= 0_{\mathbb{R}^3} \Longrightarrow \begin{cases} \alpha - \gamma = 0 \\ \beta + \gamma = 0 \\ -\alpha - a\beta - \gamma = 0 \end{cases} \\
&\Longrightarrow \begin{cases} \alpha = \beta = \gamma \\ \\ -\alpha\,(2-a) = 0 \end{cases} \\
&\Longrightarrow \begin{cases} \alpha = \beta = \gamma = 0 \\ et \\ 2 - a \neq 0 \end{cases} \\
&\Longrightarrow a \in \mathbb{R} - \{2\}\,.
\end{aligned}$$

Solution 6.4.2. *of (6.3.2) :*
Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ *be a linear application whose associated matrix in the canonical bases is*

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 2 & 1 \end{pmatrix}$$

- We have

$$f(x,y) = A \begin{pmatrix} x \\ y \end{pmatrix} = (x + 2y, -x, 2x + y)\,.$$

1) Calculate Imf *and* $\ker f$:

$$\ker f = \left\{u \in \mathbb{R}^2; f(u) = 0_{\mathbb{R}^3}\right\},$$

then

$$\begin{aligned}
f(u) &= 0_{\mathbb{R}^3} \Longrightarrow \begin{cases} x + 2y = 0 \\ -y = 0 \\ 2x + y = 0 \end{cases} \\
&\Longrightarrow (x,y) = (0,0).
\end{aligned}$$

So

$$\ker f = \{0_{\mathbb{R}^2}\}\,.$$

$$\begin{aligned} Imf &= \left\{f(u); u \in \mathbb{R}^2\right\} \\ &= \{(x+2y, -x, 2x+y); x, y \in \mathbb{R}\} \\ &= \{(x, -x, 2x) + (2y, 0, y); x, y \in \mathbb{R}\} \\ &= < \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} > . \end{aligned}$$

2) f is not bijevtive because $\dim Imf \neq \mathbb{R}^3 \neq 3$.
3) $rangA = \dim Imf = 2$.

Solution 6.4.3. *of (6.3.3) :*
Let $a \in \mathbb{R}$. We consider the matrix

$$A(a) = \begin{pmatrix} 1+a & -a \\ a & 1-a \end{pmatrix}.$$

1) Let us determine the matrix M such that $A(a) = I_2 + aM$.

$$\begin{aligned} A(a) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = I_2 + aM. \\ or \quad M &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \end{aligned}$$

2) Calculate M^2

$$M^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We deduce that :
i)

$$\begin{aligned} (A(2))^{100} &= (I_2 + 2M)^{100} \\ &= I_2^{100} + 100 \times I_2^{99} \times (2M) \\ &= I_2 + 200M \\ &= A(200). \end{aligned}$$

ii)

$$\begin{aligned} & A(1) \times A(2) \times A(3) \times .... \times A(100) \\ = \; & (I_2 + M) \times (I_2 + 2M) \times (I_2 + 3M) \times .... \times (I_2 + 100M) \\ = \; & I_2 + (1 + 2 + 3 + ... + 100) M \\ = \; & A(5050). \end{aligned}$$

6.5 Synthesis exam 2022

Exercise 6.5.1. *(The questions are independent) :*

1. *Let A, B be two square matrices of the same dimensions. If A and B are invertible, is $A + B$ invertible ? Justify.*
2. *Determine the form of a non-zero linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$.*

Exercise 6.5.2. *:*
We consider the linear application f from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose matrix in the canonical basis $\{e_1, e_2\}$ is $A = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}$.

1. *Specify the vectors $f(e_1)$ and $f(e_2)$. Specify f^2.*
2. *A^2 is it invertible ? Deduce whether f^2 is bijective.*
3. *Let $a, b \in \mathbb{R}$, determine a and b so that $f(ae_1 + be_2) = e_1 + e_2$.*

Exercise 6.5.3. *:*
We consider the linear application $g : \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix in the canonical basis B of $\mathbb{R}^3$ is

$$M = \begin{pmatrix} 5 & 3 & 6 \\ 0 & -7 & -5 \\ 0 & 3 & 1 \end{pmatrix}$$

1. *Determine the application g.*
2. *Calculate the characteristic polynomial of M, deduce that M is invertible.*
3. *Determine c and d such that $M^3 + M^2 = cM + dI_3$ then calculate M^{-1}.*
4. *Determine the eigenvalues and eigenvectors of M.*
5. *M is it diagonalizable ? If yes, give the transition matrix P from the canonical base B to the new base B'.*

Exercise 6.5.4. *:*
We consider the following (S_m) system :

$$(S_m) \begin{cases} (m-1)x + y + z = 1 \\ x + (m-1)y + z = 1 \\ x + y + (m-1)z = 1 \end{cases}, \quad m \in \mathbb{R}$$

1. *Determine the values of m so that the system admits a unique solution.*
2. *Solve the system (S_{-2}) (for $m = -2$).*

6.6 Correction of synthesis exam 2022

Solution 6.6.1. *of (6.5.1) :*

1. *Let A, B be two square matrices of the same dimensions.*
   If A and B are invertible, then $A+B$ is not invertible because : If we take for example
   $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ *et* $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *alors* $A+B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. *We notice that A and B are invertible (*$\det(A) = -1 \neq 0$ *and* $\det(B) = -1 \neq 0$*) but $A+B$ is not not invertible (*$\det(A+B) = 0$*).*
2. *The form of a non-zero linear application $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is defined for all $u = (x, y)$ by :*
$$f(u) = (ax + by, a'x + b'y, a''x + b''y).$$

Solution 6.6.2. *of (6.5.2) :*

We consider the linear application f from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose matrix in the canonical basis $\{e_1, e_2\}$ is $A = \begin{pmatrix} -2 & 1 \\ 4 & -1 \end{pmatrix}$.

1. *Let us specify the vectors $f(e_1)$ and $f(e_2)$:*
$$f(e_1) = -2e_1 + 4e_2 = (-2, 4).$$
$$f(e_2) = e_1 - e_2 = (1, -1).$$
   - Let us specify the application f^2 :
   The application $f^2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined for all $u = (x, y)$ by :
$$f^2(u) = (f \circ f)(u) = A_{mat(f)} \cdot A_{mat(f)} \cdot u = A^2 \cdot u = \begin{pmatrix} 8 & -3 \\ -12 & 5 \end{pmatrix} \cdot u$$
$$f^2(x, y) = \begin{pmatrix} 8 & -3 \\ -12 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (8x - 3y, -12x + 5y).$$
2. *A^2 is invertible because* $\begin{vmatrix} 8 & -3 \\ -12 & 5 \end{vmatrix} = 40 - 36 = 4 \neq 0.$
   Since $A^2_{mat(f^2)}$ is invertible, then f^2 is bijective.

3. *Let us determine* $a, b \in \mathbb{R}$ *so that* $f(ae_1 + be_2) = e_1 + e_2$ *:*

$$\begin{aligned} f(ae_1 + be_2) = e_1 + e_2 &\Leftrightarrow af(e_1) + bf(e_2) = e_1 + e_2 \text{ because } f \text{ is linear.} \\ &\Leftrightarrow a(-2e_1 + 4e_2) + b(e_1 - e_2) = e_1 + e_2 \\ &\Leftrightarrow (-2a + b)e_1 + (4a - b)e_2 = e_1 + e_2 \\ &\Rightarrow \begin{cases} -2a + b = 1 \\ 4a - b = 1 \end{cases} \\ &\Rightarrow \begin{cases} a = 1 \\ b = 3 \end{cases} \end{aligned}$$

Solution 6.6.3. *of (6.5.3) :*

$$M = \begin{pmatrix} 5 & 3 & 6 \\ 0 & -7 & -5 \\ 0 & 3 & 1 \end{pmatrix}$$

1. *Let us determine the linear map* g *:*
The linear application $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *is defined for all* $u = (x, y, z)$ *by :*

$$g(u) = M \cdot u = (5x + 3y + 6z, -7y - 5z, 3y + z).$$

2. *Let's calculate the characteristic polynomial* $P_M(\lambda)$ *:*

$$\begin{aligned} P_M(\lambda) = \begin{vmatrix} 5 - \lambda & 3 & 6 \\ 0 & -7 - \lambda & -5 \\ 0 & 3 & 1 - \lambda \end{vmatrix} &= (\lambda - 5)(\lambda + 2)(\lambda + 4) \\ &= -\lambda^3 - \lambda^2 + 22\lambda + 40. \end{aligned}$$

Since $P_M(0) = 40 \neq 0$*, then* M *is invertible (*M^{-1} *exists).*

3. *Let us determine* c *and* d *such that* $M^3 + M^2 = cM + dI_3$ *:*
According to the Cayley-Hamilton theorem, the characteristic polynomial is a canceling polynomial, that is to say $P_M(M) = 0$*. Then*

$$P_M(M) = -M^3 - M^2 + 22M + 40I_3 = 0$$

So $M^3 + M^2 = 22M + 40I_3$*, hence* $c = 22$ *and* $d = 40$*.*

-Let's calculate M^{-1} :

$$\begin{aligned} P_M(M) = -M^3 - M^2 + 22M + 40I_3 = 0 &\Rightarrow M^3 + M^2 - 22M = 40I_3 \\ &\Rightarrow M(M^2 + M - 22I_3) = 40I_3 \\ &\Rightarrow M^{-1}M(M^2 + M - 22I_3) = 40M^{-1}I_3 \\ &\Rightarrow I_3(M^2 + M - 22I_3) = 40M^{-1} \\ &\Rightarrow M^2 + M - 22I_3 = 40M^{-1} \\ &\Rightarrow M^{-1} = \frac{1}{40}(M^2 + M - 22I_3). \end{aligned}$$

We have : $M^2 = M \cdot M = \begin{pmatrix} 25 & 12 & 21 \\ 0 & 34 & 30 \\ 0 & -18 & -14 \end{pmatrix}$, *then*

$$\begin{aligned} M^{-1} &= \frac{1}{40}\left[\begin{pmatrix} 25 & 12 & 21 \\ 0 & 34 & 30 \\ 0 & -18 & -14 \end{pmatrix} + \begin{pmatrix} 5 & 3 & 6 \\ 0 & -7 & -5 \\ 0 & 3 & 1 \end{pmatrix} - 22\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \\ &= \frac{1}{40}\begin{pmatrix} 8 & 15 & 27 \\ 0 & 5 & 25 \\ 0 & -15 & -35 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{5} & \frac{3}{8} & \frac{27}{40} \\ 0 & \frac{1}{8} & \frac{5}{8} \\ 0 & \frac{-3}{8} & \frac{-7}{8} \end{pmatrix}. \end{aligned}$$

4. The eigenvalues of M are the roots of $P_M(\lambda) = 0$, then
$\lambda_1 = 5$, $\lambda_2 = -2$ *and* $\lambda_3 = -4$

-The associated eigenvectors :

*** For** $\lambda_1 = 5$

$$E_{\lambda_1} = \{v \in \mathbb{R}^3; Mv = \lambda_1 v = 5v\}.$$

$$Mv = 5v \Leftrightarrow \begin{cases} 5x + 3y + 6z = 5x \\ -7y - 5z = 5y \\ 3y + z = 5z \end{cases} \Rightarrow \begin{cases} 3y + 6z = 0 \\ 12y - 5z = 0 \\ 3y - 4z = 0 \end{cases} \Rightarrow \begin{cases} y = z = 0 \\ x \in \mathbb{R}. \end{cases}$$

$$E_{\lambda_1} = \left\{\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}; x \in \mathbb{R}\right\} = \left\{x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; x \in \mathbb{R}\right\} = \langle\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\rangle.$$

Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ **is an eigenvector associated with** $\lambda_1 = 5$.

*** For** $\lambda_2 = -2$

$$E_{\lambda_2} = \{v \in \mathbb{R}^3; Mv = \lambda_2 v = -2v\}.$$

$$Mv = -2v \Leftrightarrow \begin{cases} 5x+3y+6z = -2x \\ -7y-5z = -2y \\ 3y+z = -2z \end{cases} \Rightarrow \begin{cases} 7x+3y+6z = 0 \\ -5y-5z = 0 \\ 3y+3z = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{-3}{7}z \\ y = -z. \end{cases}$$

$$E_{\lambda_2} = \left\{ \begin{pmatrix} \frac{-3}{7}z \\ -z \\ z \end{pmatrix}; z \in \mathbb{R} \right\} = \left\{ z \begin{pmatrix} \frac{-3}{7} \\ -1 \\ 1 \end{pmatrix}; x \in \mathbb{R} \right\} = \langle \begin{pmatrix} \frac{-3}{7} \\ -1 \\ 1 \end{pmatrix} \rangle.$$

Let $v_2 = \begin{pmatrix} \frac{-3}{7} \\ -1 \\ 1 \end{pmatrix}$ **or** $v_2 = \begin{pmatrix} 3 \\ 7 \\ -7 \end{pmatrix}$ **is an eigenvector associated with** $\lambda_2 = -2$.

*** For** $\lambda_3 = -4$

$$E_{\lambda_3} = \{v \in \mathbb{R}^3; Mv = \lambda_3 v = -4v\}.$$

$$Mv = -4v \Leftrightarrow \begin{cases} 5x+3y+6z = -4x \\ -7y-5z = -4y \\ 3y+z = -4z \end{cases} \Rightarrow \begin{cases} 9x+3y+6z = 0 \\ -3y-5z = 0 \\ 3y+5z = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{-1}{9}z \\ y = \frac{-5}{3}z. \end{cases}$$

$$E_{\lambda_3} = \left\{ \begin{pmatrix} \frac{-1}{9}z \\ \frac{-5}{3}z \\ z \end{pmatrix}; z \in \mathbb{R} \right\} = \left\{ z \begin{pmatrix} \frac{-1}{9} \\ \frac{-5}{3} \\ 1 \end{pmatrix}; x \in \mathbb{R} \right\} = \langle \begin{pmatrix} \frac{-1}{9} \\ \frac{-5}{3} \\ 1 \end{pmatrix} \rangle.$$

Let $v_3 = \begin{pmatrix} \frac{-1}{9} \\ \frac{-5}{3} \\ 1 \end{pmatrix}$ **ou** $v_3 = \begin{pmatrix} 1 \\ 15 \\ -9 \end{pmatrix}$ **is an eigenvector associated with** $\lambda_3 = -4$.

5. **The matrix** M **is diagonalizable because** M **admits three simple eigenvalues or** $dim(E_{\lambda_1}) + dim(E_{\lambda_2}) + dim(E_{\lambda_3}) = 3 = dim(\mathbb{R}^3)$**.**
So, $\exists D$, $\exists P$ **such that :** $D = P^{-1} \cdot M \cdot P$

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

The transition matrix from B **to the new base** B' **is**
$P = \begin{pmatrix} 1 & \frac{-3}{7} & \frac{-1}{9} \\ 0 & -1 & \frac{-5}{3} \\ 0 & 1 & 1 \end{pmatrix}$ **or** $P = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 7 & 15 \\ 0 & -7 & -9 \end{pmatrix}$**.**
$B' = \{v_1, v_2, v_3\}$ **is the new basis of** $\mathbb{R}^3$.

Solution 6.6.4. *of (6.5.4) :*

1. *Let us determine the values of* m *so that the system* (S_m) *admits a unique solution :*
- Matrix writing of (S_m) *:*

$$(S_m) \Leftrightarrow \begin{pmatrix} m-1 & 1 & 1 \\ 1 & m-1 & 1 \\ 1 & 1 & m-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow A_m \cdot X = B.$$

The system (S_m) *admits a single solution iff* $\det(A_m) \neq 0$.

$$\det(A_m) = \begin{vmatrix} m-1 & 1 & 1 \\ 1 & m-1 & 1 \\ 1 & 1 & m-1 \end{vmatrix} = (m+1)(m-2)^2.$$

So $\det(A_m) \neq 0$ *iff :* $m \neq -1$ *and* $m \neq 2$ *i.e* $m \in \mathbb{R} - \{-1, 2\}$.

2. *We will solve the system* (S_{-2}) *(for* $m = -2$*) :*

$$(S_{-2}) \begin{cases} -3x + y + z = 1 \\ x - 3y + z = 1 \\ x + y - 3z = 1 \end{cases}$$

$$(S_{-2}) \Leftrightarrow \begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Leftrightarrow A_{-2} \cdot X = B.$$

Since $\det(A_m) = (m+1)(m-2)^2$ *and* $m = -2 \in \mathbb{R} - \{-1, 2\}$*, then* $\det(A_{-2}) = -16$ *and the system* (S_{-2}) *admits a unique solution.*

- Method 1 : Matrix Inversion

The solution X **of** (S_{-2}) **is given by** $X = A_{-2}^{-1} \cdot B$

$$A_{-2}^{-1} = \frac{1}{-16} \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix}.$$

Finally, $X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-1}{16} \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$

- Method 2 : Cramer Method

$$x = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{vmatrix}}{-16} = \frac{16}{-16} = -1.$$

$$y = \frac{\begin{vmatrix} -3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -3 \end{vmatrix}}{-16} = \frac{16}{-16} = -1.$$

$$z = \frac{\begin{vmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{-16} = \frac{16}{-16} = -1.$$

The solution X of (S) is $X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$.

6.7 Replacement exam 2022

Exercise 6.7.1. *:*

Let the linear application $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined for all $u = (x, y, z)$ by :

$$f(u) = (-3x - y + z, 8x + 3y - 2z, -4x - y + 2z).$$

1. *Calculate $Ker(f)$ and deduce $dimKer(f)$.*
2. *Give the basis and dimension of $Im(f)$.*
3. *f is it injective ? Is it surjective ?*
4. *Determine the matrix M associated with f in the canonical basis of $\mathbb{R}^3$.*
5. *M is it invertible ? What can we deduce ?*

Exercise 6.7.2. *:*
Let the matrix

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & -2 & 1 \\ -3 & 3 & 0 \end{pmatrix}$$

1. *Calculate the characteristic polynomial of A.*
2. *Determine the eigenvalues and eigenvectors of A.*
3. *A is it diagonalizable ? If yes, specify the transition matrix P from the canonical basis to the new basis, then calculate P^{-1}.*
4. *Give the matrix T in the new base.*

6.8 Synthesis exam 2021

Exercise 6.8.1. *:*
We consider the linear application $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by :

$$f(x, y, z) = (-x + z, x - 2y + z, -x + y + z).$$

Let $\beta = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ a canonical of $\mathbb{R}^3$.
Let $\beta' = \{v_1 = (1, 4, -1), v_2 = (1, 1, 0), v_3 = (1, 1, 2)\}$ a basis of $\mathbb{R}^3$.

1. *Calculate $Ker(f)$. Deduce $dimKer(f)$ and $dimIm(f)$*
2. *f is it injective ? Surjective ?*
3. *Determine the matrix M associated with f in β.*

4. *Write the transition matrix P from β to β'.*
5. *Determine the matrix D associated with f in the basis β'.*

Exercise 6.8.2. *:*
PART I : *Let the matrix*

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

1. *Calculate the characteristic polynomial of A.*
2. *Determine the eigenvalues and eigenvectors of A.*
3. *A is it diagonalizable ?*

PART II :

1. *Determine for which values of m, the matrix B is invertible :*

$$B = \begin{pmatrix} 1 & -1 & 2 \\ m-1 & 0 & 1 \\ -2 & 2 & 1 \end{pmatrix}, m \in \mathbb{R}.$$

2. *Solve the following system of linear equations :*

$$(S) \begin{cases} x - y + 2z = -1 \\ x + z = 1 \\ -2x + 2y + z = -3 \end{cases}.$$

Exercise 6.8.3. *:*
Let $E = \{(x, y, z, t) \in \mathbb{R}^4;\ \ x + 2z = 0 \ \ et \ \ 3y + t = 0\}$ a vector subspace of $\mathbb{R}^4$.
Let $u_1 = (1, 1, 0, 0)$; $u_2 = (1, 1, 0, 1)$; $u_3 = (1, 0, 1, 1)$ and $F =< u_1, u_2, u_3 >$.

1. *Give a basis of E and deduce $dim(E)$.*
2. *Calculate $dim(F)$. Do we have $E \oplus F = \mathbb{R}^4$?*

6.9 Correction of synthesis exam 2021

Solution 6.9.1. *of (6.8.1) :*
We consider the linear application $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by :

$$f(x, y, z) = (-x + z, x - 2y + z, -x + y + z).$$

1. *Calculons $Ker(f)$*

$$Ker(f) = \{u \in \mathbb{R}^3; f(u) = 0_{\mathbb{R}^3}\}. \ \ Then$$

$$f(u) = 0_{\mathbb{R}^3} \Leftrightarrow \begin{cases} -x + z = 0 \\ x - 2y + z = 0 \\ -x + y + z = 0 \end{cases} \Rightarrow \begin{cases} z = x \\ 2x - 2y = 0 \\ y = 0 \end{cases} \Rightarrow x = y = z = 0. \ So,$$

$Ker(f) = \{0_{\mathbb{R}^3}\} \Longrightarrow dimKer(f) = 0.$

Let us calculate $dimIm(f)$ *: According to the rank theorem we have* $dimIm(f) = dim\mathbb{R}^3 - dimKer(f) = 3 - 0 = 3.$

2. *The application* f *is injective because* $dimKer(f) = 0$ *ou* $Ker(f) = \{0_{\mathbb{R}^3}\}$.
The application f *is surjective because* $dimIm(f) = 3 = dim(\mathbb{R}^3)$ *or* $Im(f) = \mathbb{R}^3$.

3. *Let us determine* M *the matrix associated with* f *in the canonical basis* β *of* $\mathbb{R}^3$
$f(e_1) = (-1, 1, -1) = -e_1 + e_2 - e_3.$
$f(e_2) = (0, -2, 1) = 0e_1 - 2e_2 + e_3.$
$f(e_3) = (1, 1, 1) = e_1 + e_2 + e_3.$
So

$$M = Mat_\beta(f) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

4. *Let us write the transition matrix* P *from* β *to* β'
$v_1 = (1, 4, -1) = e_1 + 4e_2 - e_3.$
$v_2 = (1, 1, 0) = e_1 + e_2 + 0e_3.$
$v_3 = (1, 1, 2) = e_1 + e_2 + 2e_3.$
Therefore

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}.$$

5. *Let us determine* D *the matrix associated with* f *in the base* β' *of* $\mathbb{R}^3$.
- Method 1 :

$$D = P^{-1} \cdot M \cdot P$$

We calculate P^{-1} **:** $P^{-1} = \frac{1}{det(A)} com^t(A)$

$$\det(P) = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = -6.$$

$$com(P) = \begin{pmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & - \begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix} & + \begin{vmatrix} 4 & 1 \\ -1 & 0 \end{vmatrix} \\ - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} & + \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} & - \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} \\ + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & - \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & -9 & 1 \\ -2 & 3 & -1 \\ 0 & 3 & -3 \end{pmatrix}, \textbf{ then}$$

$$com^t(P) = \begin{pmatrix} 2 & -2 & 0 \\ -9 & 3 & 3 \\ 1 & -1 & -3 \end{pmatrix}.$$

So $P^{-1} = \frac{-1}{6}\begin{pmatrix} 2 & -2 & 0 \\ -9 & 3 & 3 \\ 1 & -1 & -3 \end{pmatrix}$.

Finally

$$D = \frac{-1}{6}\begin{pmatrix} 2 & -2 & 0 \\ -9 & 3 & 3 \\ 1 & -1 & -3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Method 2 :

$$D = Mat_{\beta'}(f) = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

With

$f(v_1) = f(1,4,-1) = (-2,-8,2) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3.$
$f(v_2) = f(1,1,0) = (-1,-1,0) = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3.$
$f(v_3) = f(1,1,2) = (1,1,2) = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3.$

$$\begin{pmatrix} -2 \\ -8 \\ 2 \end{pmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \Leftrightarrow \begin{cases} -2 = \alpha_1 + \alpha_2 + \alpha_3 \\ -8 = 4\alpha_1 + \alpha_2 + \alpha_3 \\ 2 = -\alpha_1 + 2\alpha_3 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1 = -2 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \end{cases}$$

$$\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 \Leftrightarrow \begin{cases} -1 = \beta_1 + \beta_2 + \beta_3 \\ -1 = 4\beta_1 + \beta_2 + \beta_3 \\ 0 = -\beta_1 + 2\beta_3 \end{cases}$$

$$\Rightarrow \begin{cases} \beta_1 = 0 \\ \beta_2 = -1 \\ \beta_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 \Leftrightarrow \begin{cases} 1 = \gamma_1 + \gamma_2 + \gamma_3 \\ 1 = 4\gamma_1 + \gamma_2 + \gamma_3 \\ 2 = -\gamma_1 + 2\gamma_3 \end{cases}$$

$$\Rightarrow \begin{cases} \gamma_1 = 0 \\ \gamma_2 = 0 \\ \gamma_3 = 1 \end{cases}$$

Solution 6.9.2. *of (6.8.2) :*
PART I : *Let the matrix*

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -2 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

1. *Let's calculate the characteristic polynomial* $P_A(\lambda)$ *:*

$$P_A(\lambda) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 1 & -2-\lambda & 1 \\ -1 & 1 & -1-\lambda \end{vmatrix} = (\lambda+2)(\lambda+1)^2.$$

2. The eigenvalues of A are the roots of $P_A(\lambda) = 0$, then $\lambda_1 = -2$ (single) and $\lambda_2 = -1$ (double).

- The associated eigenvectors :

For $\lambda_1 = -2$

$$E_{\lambda_1} = \{v \in \mathbb{R}^3; Av = \lambda_1 v = -2v\}.$$

$$Av = -2v \Leftrightarrow \begin{cases} -x+z = -2x \\ x-2y+z = -2y \\ -x+y-z = -2z \end{cases} \Rightarrow \begin{cases} x+z = 0 \\ x+z = 0 \\ -x+y+z = 0 \end{cases} \Rightarrow \begin{cases} z = -x \\ y = x-z = 2x. \end{cases}$$

$$E_{\lambda_1} = \left\{ \begin{pmatrix} x \\ 2x \\ -x \end{pmatrix}; x \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}; x \in \mathbb{R} \right\} = \langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \rangle.$$

Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ **an eigenvector associated with** $\lambda_1 = -2$.

For $\lambda_2 = -1$

$$E_{\lambda_2} = \{v \in \mathbb{R}^3; Av = \lambda_2 v = -v\}.$$

$$Av = -v \Leftrightarrow \begin{cases} -x+z = -x \\ x-2y+z = -y \\ -x+y-z = -z \end{cases} \Rightarrow \begin{cases} z = 0 \\ x-y+z = 0 \\ -x+y = 0 \end{cases} \Rightarrow \begin{cases} z = 0 \\ y = x. \end{cases}$$

$$E_{\lambda_2} = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}; x \in \mathbb{R} \right\} = \left\{ x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; x \in \mathbb{R} \right\} = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \rangle.$$

Let $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ **an eigenvector associated with** $\lambda_2 = -1$.

3. The matrix A is not diagonalizable because $dim(E_{\lambda_1}) + dim(E_{\lambda_2}) = 2 \neq 3 = dim(\mathbb{R}^3)$**.**

Part II :

1. Let's determine the values of m so that B is invertible :

$$B = \begin{pmatrix} 1 & -1 & 2 \\ m-1 & 0 & 1 \\ -2 & 2 & 1 \end{pmatrix}, m \in \mathbb{R}.$$

B is invertible iff $\det(B) \neq 0$.

$$\det(B) = \begin{vmatrix} 1 & -1 & 2 \\ m-1 & 0 & 1 \\ -2 & 2 & 1 \end{vmatrix} = 5m - 5.$$

Therefore, the matrix B is invertible iff : $5m - 5 \neq 0$ *i.e :* $m \neq 1$ ($m \in \mathbb{R} - \{1\}$).

2. *We will solve the following system of linear equations*

$$(S) \begin{cases} x - y + 2z = -1 \\ x + z = 1 \\ -2x + 2y + z = -3 \end{cases}.$$

- The matrix form of (S) is

$$(S) \Leftrightarrow \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ -2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} \Leftrightarrow B_2 \cdot X = b.$$

The system (S) admits a single solution iff $\det(B_2) \neq 0$.
As $\det(B_2) = 5 \cdot 2 - 5 = 5 \neq 0$, *then (S) has only one solution.*

- Method 1 : Matrix inversion

The solution X of (S) is given by $X = B_2^{-1} \cdot b$

$$B_2^{-1} = \frac{1}{5} \begin{pmatrix} -2 & 5 & -1 \\ -3 & 5 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Finally, $X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-1}{5} \begin{pmatrix} -2 & 5 & -1 \\ -3 & 5 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$

- Method 2 : Cramer's method

$$x = \frac{\begin{vmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ -3 & 2 & 1 \end{vmatrix}}{5} = \frac{10}{5} = 2.$$

$$y = \frac{\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ -2 & -3 & 1 \end{vmatrix}}{5} = \frac{5}{5} = 1.$$

$$z = \frac{\begin{vmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ -2 & 2 & -3 \end{vmatrix}}{5} = \frac{-5}{5} = -1.$$

The solution X of (S) is $X \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$

Solution 6.9.3. *of (6.8.3) :*

1. *Basis and dimension of E :*

$$\begin{aligned} E &= \{(x,y,z,t) \in \mathbb{R}^4; \quad x+2z=0 \quad et \quad 3y+t=0\} \\ &= \{(x,y,z,t) \in \mathbb{R}^4; \quad x=-2z \quad et \quad t=-3y\} \\ &= \{(-2z,y,z,-3y); \quad y,z \in \mathbb{R}\} \\ &= \{y(0,1,0,-3)+z(-2,0,1,0); \quad y,z \in \mathbb{R}\} \\ &= \langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rangle. \end{aligned}$$

We notice that the two vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -3 \end{pmatrix}$ *et* $\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ *are linearly independents, so the family* $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ *form a basis of E.*
Hence $dim(E) = 2$.

2. *We calculate $dim(F)$:*
We have $F =< u_1, u_2, u_3 >$, let us show that $\{u_1, u_2, u_3\}$ are linearly independent

$$\forall \alpha,\beta,\gamma \in \mathbb{R}; \alpha u_1+\beta u_2+\gamma u_3 = 0_{\mathbb{R}^4} \Leftrightarrow \begin{cases} \alpha+\beta+\gamma=0 \\ \alpha+\beta=0 \\ \gamma=0 \ \checkmark \\ \beta+\gamma=0 \end{cases} \Rightarrow \alpha=\beta=0.$$

So the family $\{u_1, u_2, u_3\}$ is a basis of F, hence $dim(F) = 3$.

- Direct sum of E and F :
As $dim(E) + dim(F) = 2 + 3 = 5 \neq dim(\mathbb{R}^4)$, then the sum is not direct i.e $E \oplus F \neq \mathbb{R}^4$.

6.10 Replacement exam 2021

Exercise 6.10.1. *:*
PART I : *Let the application $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by :*

$$f(x,y,z) = (-3x+y+z, x-3y+z, x+y-3z).$$

1. *Show that f is linear.*
2. *Calculate $Ker(f)$, $Im(f)$ and give their dimensions.*

3. *f is it injective ? Surjective ?*
4. *Give a basis of $Im(f)$.*

Exercise 6.10.2. *:*
We consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$$

1. *A is it invertible ?, if so calculate A^{-1}.*
2. *Solve the following system of linear equations by using two methods :*

$$(S) \begin{cases} x - 2y + z = 1 \\ -2x + 2y + 3z = 0 \\ y - z = -1 \end{cases}$$

6.11 Interrogation exam 2020

Exercise 6.11.1. *(Course Questions)*
Part I : *Let E, F and G be three vector spaces and $f : E \longrightarrow F$, $g : F \longrightarrow G$ be two linear applications.*

1. *Show that $Im(f)$ is a vector subspace of F.*
2. *Show that $g \circ f$ is linear.*

Part II : *(The questions are independent)*

1. *We designate by E a $\mathbb{R}$-vector space of finite dimension.*
   "If the vectors u, v, w are free in pairs, then the family $\{u, v, w\}$ is free" is it true or false ? Justify.
2. *Determine the form of a non-zero linear application from $\mathbb{R}^2$ to $\mathbb{R}^3$.*
3. *Describe the vector subspaces of $\mathbb{R}^2$.*

Exercise 6.11.2. *:*
Let the application $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by :

$$f(x, y, z) = (-3x + y + z, x - 3y + z, x + y - 3z).$$

1. *Show that f is linear.*
2. *Calculate $Ker(f)$, $Im(f)$ and give their dimensions.*
3. *f is it injective ? Surjective ?*
4. *Do we have $ker(f) \oplus Im(f) = \mathbb{R}^3$?*

Exercise 6.11.3. *:*
Let $E = \{(x, y, z, t) \in \mathbb{R}^4;\ \ x + 2z = 0 \ \ et \ \ 3y + t = 0\}$.
Let $u_1 = (1, 1, 1, 1)$; $u_2 = (1, -1, 1, -1)$; $u_3 = (1, 0, 1, 0)$ and $F =< u_1, u_2, u_3 >$.

1. *Show that E is a vector subspace of $\mathbb{R}^4$.*
2. *Give a basis of E and deduce $dim(E)$.*
3. *Calculate $dim(F)$ and $dim(E \cap F)$.*
4. *Do we have $E \oplus F = \mathbb{R}^4$?*

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Exercise 6.12.1. *:*
Let $E = \{(x, y, z) \in \mathbb{R}^3;\ \ x - 2y + z = 0\ \}$ be a vector subspace of $\mathbb{R}^3$.
Let $u_1 = (1, 1, 1)\,;\ u_2 = (1, -1, 1)$ and $F =< u_1, u_2 >$.

1. *Give a basis of E and deduce $dim(E)$.*
2. *Calculate $dim(F)$. Do we have $E \oplus F = \mathbb{R}^3$?*
3. *Calculate $dim(E \cap F)$.*

Exercise 6.12.2. *:*
Let the linear map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined for all $u = (x, y, z)$ by :

$$f(u) = (x - y, 2x + y - z, y + z).$$

1. *Calculate $Ker(f)$, $Im(f)$ and give their dimensions.*
2. *f is it injective ? Surjective ?*
3. *Do we have $ker(f) \oplus Im(f) = \mathbb{R}^3$?*

Exercise 6.12.3. *:*
Consider the following matrices :

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -3 \\ 0 & -1 & 2 \\ 3 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 3 \\ 0 & 1 \\ 2 & 2 \end{pmatrix}.$$

1. *Calculate the possible products of these matrices.*
2. *B is it invertible ?, if so calculate B^{-1}.*

6.13 Correction of synthesis exam 2020

Solution 6.13.1. *of (6.12.1) :*

1. *Basis and dimension of E :*

$$\begin{aligned} E &= \{(x,y,z)\in\mathbb{R}^3;\quad x-2y+z=0\} \\ &= \{(x,y,z)\in\mathbb{R}^3;\quad x=2y-z\} \\ &= \{(2y-z,y,z);\quad y,z\in\mathbb{R}\} \\ &= \{y(2,1,0)+z(-1,0,1);\quad y,z\in\mathbb{R}\} \\ &= \langle\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\rangle. \end{aligned}$$

We notice that the two vectors $\begin{pmatrix}2\\1\\0\end{pmatrix}$ *et* $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ *are linearly independent, so the family* $\left\{\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ *form a basis of* E.
Hense $dim(E)=2$.

2. *We calculate* $dim(F)$ *: We have* $F=<u_1,u_2>$, *so the family* $\{u_1,u_2\}$ *is generator, also it is linearly independent. So the family* $\{u_1,u_2\}$ *is a basis of* F, *hence* $dim(F)=2$.
- Direct sum of E *and* F *:*
As $dim(E)+dim(F)=2+2=4\neq dim(\mathbb{R}^3)$, *then the sum is not direct i.e* $E\oplus F\neq\mathbb{R}^3$.

3. $dim(E\cap F)=1$.

Solution 6.13.2. *of (6.12.2) :*

$$f(u)=(x-y,2x+y-z,y+z).$$

1. *Let's calculate* $Ker(f)$

$$Ker(f)=\{u\in\mathbb{R}^3;f(u)=0_{\mathbb{R}^3}\}.\quad Then$$

$$f(u)=0_{\mathbb{R}^3}\Leftrightarrow\begin{cases}x-y=0\\2x+y-z=0\\y+z=0\end{cases}\Rightarrow\begin{cases}y=x\\3y=0\\y=-z\end{cases}\Rightarrow x=y=z=0.\ So,$$

$Ker(f)=\{0_{\mathbb{R}^3}\}\Longrightarrow dimKer(f)=0$.

Let us calculate $dimIm(f)$ *: According to the rank theorem we have* $dimIm(f)=dim\mathbb{R}^3-dimKer(f)=3-0=3$.

2. *The application* f *is injective because* $dimKer(f)=0$ *ou* $Ker(f)=\{0_{\mathbb{R}^3}\}$.
The application f *is surjective because* $dimIm(f)=3=dim(\mathbb{R}^3)$ *or* $Im(f)=\mathbb{R}^3$.

3. *The sum is direct,* $ker(f) \oplus Im(f) = \mathbb{R}^3$.

Solution 6.13.3. *of (6.12.3) :*

1. *The possible products are* CA, BC, A^2 *and* B^2.
$$CA = \begin{pmatrix} 5 & -12 \\ -1 & 0 \\ 2 & -6 \end{pmatrix}, \quad BC = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 10 & 7 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 3 & -6 \\ -2 & 3 \end{pmatrix},$$
$$B^2 = \begin{pmatrix} -5 & 1 & -1 \\ 6 & 1 & -4 \\ 3 & 3 & -8 \end{pmatrix}.$$
2. B *is inversible because*
$$\det B = \begin{vmatrix} 2 & 1 & -3 \\ 0 & -1 & 2 \\ 3 & 0 & -1 \end{vmatrix} = -1 \neq 0.$$

Let's calculate B^{-1} *:*

$B^{-1} = \frac{1}{det(B)} adj\,(B)$
We have $\det(B) = -1$.

$$adj(B) = \begin{pmatrix} +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} & +\begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -3 \\ 0 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & -31 \\ 3 & 0 \end{vmatrix} \\ +\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} \end{pmatrix}^T$$
$$= \begin{pmatrix} 1 & 1 & -1 \\ 6 & 7 & -4 \\ 3 & 3 & -2 \end{pmatrix}$$

Then
$$B^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & 1 & -1 \\ 6 & 7 & -4 \\ 3 & 3 & -2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ -6 & -7 & 4 \\ -3 & -3 & 2 \end{pmatrix}.$$

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